

PL Equivariant Surgery and Invariant Decompositions of 3-Manifolds

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INTRODUCTION

Using normal surface theory $[H_1, J_2]$, we introduce the notion of least weight normal surfaces. The weight of a normal surface is a nonnegative integer invariant of the normal isotopy class of the surface. If we focus on a particular class of normal surfaces and choose representatives which minimize the weight over the class, then we have least weight normal surfaces. It is remarkable how these least weight normal surfaces exhibit many of the same useful properties as least area (minimal) surfaces. They provide a piecewise linear (PL) environment to obtain the recent topological results coming from the analysis and geometry of least area surfaces.

In an impressive series of papers Meeks and Yau $[M-Y_1, M-Y_2, M-Y_3, M-Y_4, M-Y_5]$, Meeks, Simon, and Yau $[M-S-Y]$, Scott $[S]$, Meeks and Scott $[M-S]$, and Freedman, Hass, and Scott $[F-H-S]$ introduced the analysis and geometry of least area surfaces into the study of topological questions about 3-manifolds. The consequences have led to the resolution of many outstanding problems in the topology of 3-manifolds. Since most of the problems being solved with these new techniques are topological in nature, it has been felt that there should be a more topological approach to the proofs of these results. Besides, the bulk of the work in the theory of

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least area surfaces is often the long and tedious arguments from analysis and geometry establishing existence of appropriate least area surfaces.

In our approach we use only the methods of piecewise linear (PL) topology. The problems of existence, which hamper the analytic approach, are virtually missing in the PL approach. Most were established in 1928 by H. Kneser [K]. But, there is more benefit in the PL approach than the fact the solutions to the existence problems are classical. The PL existence results are easy to prove and can be done in a very short space. (Since this is one of the attractions of the PL approach, we include the necessary existence theorems, with proof, in Section 2.) Also least weight normal surfaces can be explicitly constructed via algorithms, unlike the analytic case.

In this paper, we concentrate on the investigation of group actions on 3-manifolds. The use of least weight surfaces, in the framework of normal surfaces, allows us to make equivariant moves (with respect to group actions), which long had eluded the PL approach.

For example, in Section 4 we give a PL proof of The Equivariant Sphere Theorem $[M-Y_2]$. A special case of which states for M a 3-manifold and G a properly discontinuous group of diffeomorphisms of M , if $\pi_2(M)$ is not trivial, then there is a G -equivariant, embedded 2-sphere or two-sided projective plane in M , which represents a nontrivial class in $\pi_2(M)$. (See Section 4 for the necessary definitions.) A natural consequence of this is the original Sphere Theorem $[P_1]$. The version of Papakyriakopoulos had a restricted hypothesis which avoided technical difficulties in a case where torsion elements appeared. Later J. H. C. Whitehead [W] was able to overcome these technical problems and remove the restricted hypothesis. The version of the Sphere Theorem allowing for two-sided projective planes is a refinement due to Epstein [Ep]. All of these approaches use the so-called "tower-construction" and are independent of the Loop Theorem $[P_2]$. It has always been curious that the various proofs of the Sphere Theorem $[W, M-Y_2, St_2]$ are much more difficult than the proofs of the Loop Theorem/Dehn's Lemma, even though most use the same general techniques. Equally as curious is the fact that it seemed impossible to meaningfully use the Loop Theorem in gaining a proof of the Sphere Theorem. Our proof of the Equivariant Sphere Theorem does not use the tower construction and it gives the satisfaction of using the Loop Theorem in an essential way. Furthermore, our proofs of the Equivariant Sphere Theorem and the Equivariant Loop Theorem use identical techniques. We state the Equivariant Loop Theorem in Section 4 ($[M-Y_2]$, also see $[M-Y_5]$). The special case of the Equivariant Loop Theorem for involutions was shown in $[G-L]$ and independently, in $[K-T]$. In $[Ed]$ a combinatorial proof of the Equivariant Dehn's Lemma is given.

We also obtain a proof that any covering of a P^2 -irreducible 3-manifold is P^2 -irreducible $[M-S-Y]$. It is sufficient to show that the universal cover-

ing of such a 3-manifold is P^2 -irreducible. This can be proved using exactly the same arguments that we use in proving the Equivariant Sphere Theorem.

In Section 5 we refine our considerations to finding G -invariant collections of surfaces. In general, if one has a G -equivariant collection, then by considering all G -orbits of the collection one obtains a G -invariant collection. Quite often such a G -invariant collection has lots of redundancies. We show that in the case of certain prime decompositions of 3-manifolds and in the case of the characteristic decomposition of a ∂ -irreducible Haken manifold, these redundancies can be avoided. Specifically, if M is a compact 3-manifold, G is a finite group of diffeomorphisms of M , and M admits a prime decomposition without closed 1-handles, then M admits a G -invariant, prime decomposition. This reduces the study of finite group actions on such manifolds to group actions on their prime factors. This result improves on a similar result proved in [M-Y₁].

In the case that M is a ∂ -irreducible Haken manifold and G is a finite group of diffeomorphisms of M , then the characteristic submanifold of M can be chosen to be G -invariant or M is a torus bundle over S^1 and the characteristic submanifold of M is a neighborhood of the fiber. This theorem was first proved, using least area annuli and tori, in the paper [M-S].

We have organized the paper in the following way. In Section 1 we review the part of normal surface theory which is relevant to this paper. We define normal surfaces and the concept of geometric addition of normal surfaces. Several combinatorial properties of normal surfaces are reviewed, with particular emphasis on the fact that normal surface theory takes place in the 2-skeleton of a triangulation of the 3-manifold.

In Section 2 we define the weight of a normal surface and give several classical results on the existence of (least weight) normal surfaces. In particular, if a 3-manifold M contains an embedded, essential 2-sphere (e.g., an embedded 2-sphere which does not bound a 3-cell) then for any triangulation of the 3-manifold M , there is an embedded, essential, normal 2-sphere. Hence, for any triangulation of such a 3-manifold M , there exist essential, least weight normal 2-spheres. Similarly, if a 3-manifold M contains an incompressible and ∂ -incompressible surface F , then for any triangulation of M , there is an incompressible and ∂ -incompressible, least weight normal surface topologically equivalent to F . In fact, if M is irreducible and ∂ -irreducible, then for any triangulation of M each isotopy class of incompressible and ∂ -incompressible surface is represented by a least weight normal surface.

In Section 3 we establish new material which makes the PL approach work. The idea comes from the theory of least area surfaces. Namely, when working with least area surfaces, a general rule is, if two least area surfaces

intersect, then, in some sense, the intersection is as uncomplicated as possible. Now, normal surfaces have a lot of freedom of movement in their normal isotopy class and so two such surfaces are expected to have (and do have) lots of intersections. However, if two least weight normal surfaces intersect, then, in some sense, the intersection is reasonably simple and can be managed. In particular, if two essential, least weight normal 2-spheres meet transversely, then there exists curves of intersection, which are innermost on both. To have such curves of intersection, which are innermost on both surfaces, is a totally unexpected result. Even more surprisingly, every curve of intersection has a property which is special in normal surface theory, they are regular curves of intersection. Since curves of intersection are regular curves, there is a unique way to make exchanges (regular exchanges) along the curves. On one hand, these regular exchanges provide a unique choice for making exchanges and allow us to define equivariant operations. On the other hand, these regular exchanges correspond to the geometric addition of normal surface theory and provide us with knowledge of the end result of these equivariant operations.

In Section 4 we state and prove the major results of this paper. The main tools, of course, are the intersection properties of normal surfaces developed in Section 3; however, we do introduce a new concept for the complexity of the intersections of surfaces, relative to the action of a properly discontinuous group of simplicial homeomorphisms. This complexity is defined for any normal surface and is a nonnegative integer invariant of the 1-skeleton (of the induced cell structure) of the normal surface and the group.

Finally, in Section 5, we refine the results of Section 4 to the notion of G -invariance as opposed to G -equivariance for finite group actions on compact 3-manifolds. In a subsequent work, we will build on the idea of least weight normal surfaces to give a PL analogue of minimal surfaces in 3-manifolds, which enables us to study the case of singular surfaces. The terms and notation used throughout the paper are standard. Basic definitions can be found in $[J_2]$ or $[H]$.

We have been informed by P. Scott that M. J. Dunwoody $[D]$ has obtained a proof of our Theorem 4.4 using combinatorial methods.

1. REVIEW OF NORMAL SURFACES

Our approach to the theory of normal surfaces is via triangulations of the 3-manifolds under consideration (see $[J_2]$ for a detailed exposition of Normal Surface Theory). Hence, throughout this paper a 3-manifold M will come equipped with a triangulation \mathcal{T} . An isotopy of M is called a *normal isotopy* (with respect to \mathcal{T}) if it leaves the various simplices of \mathcal{T}

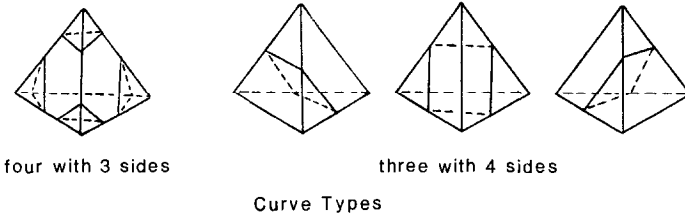


FIGURE 1

invariant. A normal surface in M (a normal surface with respect to \mathcal{T}) is simply a 2-manifold in M which has particularly nice intersection properties relative to the tetrahedra of \mathcal{T} . In this section we give a formal definition of a normal surface and state most of the properties of normal surfaces needed in later sections.

Let Δ be a tetrahedron in the triangulation \mathcal{T} of the 3-manifold M . A (simple) closed curve in $\partial\Delta$, the boundary of Δ , is called a *curve type* of \mathcal{T} if it meets the faces of Δ in straight spanning arcs and meets any given face at most once. A tetrahedron Δ of \mathcal{T} has, up to normal isotopy, precisely seven curve types of \mathcal{T} . There are four curve types with three sides and three curve types with four sides (see Fig. 1).

If α is a curve type in $\partial\Delta$, the boundary of Δ , and p is a point in $\text{int}(\Delta)$, the interior of Δ , then the join, $p * \alpha$, is called a *disk type* of \mathcal{T} . There are, of course, several ways of filling in a curve type to obtain a disk type and our method depends upon the point p . However, a normal isotopy class of curve types determines a normal isotopy class of disk types. Hence, a tetrahedron Δ has up to normal isotopy precisely seven disk types of \mathcal{T} ; and if t is the total number of tetrahedra of \mathcal{T} , then \mathcal{T} has $7t$ normal isotopy classes of disk types. A properly embedded surface F in a 3-manifold M is a *normal surface* (with respect to a triangulation \mathcal{T} of M) if F meets each tetrahedron of \mathcal{T} in a (necessarily pairwise disjoint) collection of disk types of \mathcal{T} .

A normal surface is determined, up to normal isotopy, by the collection of curve types in which it meets the boundaries of the various tetrahedra of \mathcal{T} . In fact, suppose $\mathcal{C}_1, \dots, \mathcal{C}_n$ is any ordering of the normal isotopy classes of curve types of \mathcal{T} and F is a normal surface. Then F determines (and is itself determined by) an n -tuple of nonnegative integers (x_1, \dots, x_n) , where x_i denotes the number of representatives of the curve type \mathcal{C}_i ($1 \leq i \leq n$) in the total collection of curve types in which F meets the boundaries of the tetrahedra of \mathcal{T} .

Once an ordering of the normal isotopy classes of curve types of \mathcal{T} is made, then the normal isotopy class of a normal surface corresponds uniquely to an n -tuple of nonnegative integers. Conversely, we could begin

with an n -tuple of nonnegative integers and try to build a normal surface in M , which corresponds to such an n -tuple. This is possible only if we subject the n -tuple to two constraints.

The first constraint is that a normal surface must meet a tetrahedron in pairwise disjoint disk types. Hence, the positive entries of the n -tuple must correspond to normal isotopy classes of curve types which can be realized by piecewise disjoint curve types on the boundaries of the various tetrahedron of \mathcal{T} .

The second constraint is a matching constraint and arises in trying to match the edges of disk types through the incident faces of tetrahedra. Namely, if F intersects a face σ of a tetrahedron Δ in p representatives of a particular arc type and σ is incident in the triangulation \mathcal{T} with a face σ' of a tetrahedron Δ' , then F intersects the face σ' in p representatives of the corresponding arc type in σ' . Now, for any tetrahedron Δ in \mathcal{T} and any face σ of Δ there are two normal isotopy classes of disk types in Δ which meet σ in one of the normal isotopy classes of the arc types in σ . So, the second constraint can be described by a system of linear equations, each of the form

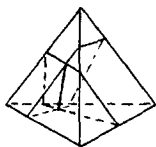
$$x_i + x_j = x_k + x_l. \quad (*)$$

We have one such normal equation for each possible incident pair of normal isotopy classes of arcs in the faces of \mathcal{T} . If t is the number of tetrahedra of \mathcal{T} , then there are at most $6t$ such normal equations. The maximum number of $6t$ equations is obtained when M is without boundary; however, even in this case the equations are not in general independent.

If F is a normal surface and the normal isotopy class of F corresponds to the n -tuple (x_1, \dots, x_n) , as above, then (x_1, \dots, x_n) is a solution to the system of normal equations (*). On the other hand, any solution (x_1, \dots, x_n) to (*), where x_i is a nonnegative integer, corresponds to a normal isotopy class of a normal surface, subject to the first constraint above. If this particular constraint is satisfied (namely, the nonzero entries in (x_1, \dots, x_n) correspond to normal isotopy classes which can be represented by pairwise disjoint curve types), the solution (x_1, \dots, x_n) is said to be *realizable*.

If F is a normal surface (with respect to the triangulation \mathcal{T}), then F inherits a natural cell structure from \mathcal{T} . The cells are the components of the intersections of F with the various tetrahedra of \mathcal{T} . Therefore, each cell in this induced cell structure on F is either 3-sided or 4-sided. We shall refer to this cell structure on a normal surface F as the *induced cell structure*.

In this paper, we are particularly interested in the intersection of normal surfaces. We now introduce the preliminary concepts for understanding the intersection of normal surfaces. Let F and F' be normal surfaces meeting transversely. The components of intersection between F and F' (simple



An exceptional pairing of disk types

FIGURE 2

closed curves and spanning arcs) fall into two possible types as follows: A component C of $F \cap F'$ is made up of a union of arcs, each of which is the intersection of a disk type from F and a disk type from F' in a tetrahedron Δ of \mathcal{T} . The intersection of a disk type from F and a disk type from F' in a tetrahedron Δ can come from any of the possible pairings taken over the seven different normal isotopy classes of disk types. However, there are pairings which create exceptional problems. The exceptional pairings occur when the disk type from F has four sides and the disk type from F' has four sides and the two 4-sided disks are in distinct normal isotopy classes (see Fig. 2).

In the case that a component C of $F \cap F'$ contains an arc coming from an exceptional pairing in some tetrahedron, we say C is a *singular curve of intersection between F and F'* . Otherwise, we say C is a *regular curve of intersection between F and F'* .

Regular curves of intersection between normal surfaces have a special place in the theory and provide an algebraic environment for standard geometric “cut-and-paste” techniques. We say two normal surfaces F and F' are *compatible* if either $F \cap F' = \emptyset$ or each component of $F \cap F'$ is a regular curve. We shall now analyze the combinatorics and the geometric interpretation of the intersection of normal surfaces.

If C is a curve of $F \cap F'$, then C meets the 2-skeleton of \mathcal{T} in a finite set of points and any one of these points can be viewed as the point of intersection between two straight spanning arcs in a 2-simplex which is a face of a tetrahedron of \mathcal{T} (see Fig. 3). In this situation there are two ways to

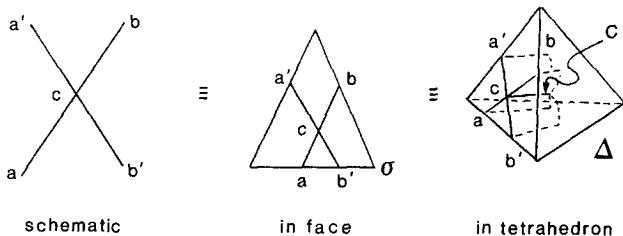


FIGURE 3

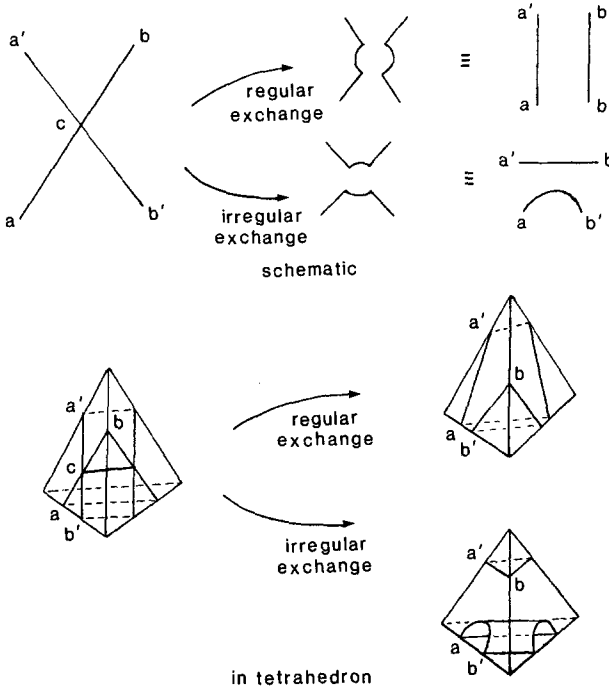


FIGURE 4

exchange the end points of these arcs and rejoin with arcs; however, a major observation is that there is a *unique* way to exchange the end points of these arcs and rejoin them with *straight* spanning arcs. This is called a *regular exchange* at the particular point of C in question (see Fig. 4).

If C is a regular curve of intersection between the normal surfaces F and F' , then C is orientation preserving in M and therefore a small neighborhood, $N(C)$, of C is a solid torus. Furthermore, when C is a regular curve, C is either orientation preserving on both F and F' or C is orientation reversing on both F and F' . In the former situation F and F' meet $N(C)$ in annuli $A \subset F$ and $A' \subset F'$; and in the latter situation F and F' meet $N(C)$ in Moebius Bands $B \subset F$ and $B' \subset F'$. No matter what the case, we have $A \cap A' = C$ or $B \cap B' = C$.

Let T be the torus boundary of $N(C)$. If C is orientation preserving on both F and F' , then T is divided into four annuli T_0, T'_0, T_1, T'_1 by F and F' ; whereas, if C is orientation reversing on both F and F' , then T is divided into two annuli T_0 and T'_0 (see Fig. 5).

As always, in such a situation there are two possible "cut-and-paste" exchanges. One is to remove A from F and A' from F' (B from F and B' from F') and add the annuli T_0 and T_1 (and add the annulus T_0). The

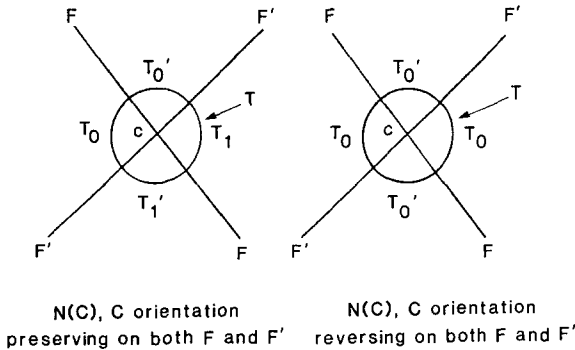


FIGURE 5

other is to remove A from F and A' from F' (B from F and B' from F') and add the annuli T'_0 and T'_1 (and add the annulus T'_0). Now, the point about considering such a “cut-and-paste” operation along a regular curve C of intersection between F and F' is that there is a unique choice for the annuli, in order to make the “cut-and-paste” exchanges, which corresponds to the unique choice for a regular exchange at each point where C meets the 2-skeleton of \mathcal{T} as defined above. One can check that this is never the case along a singular curve of intersection. We will call such a unique “cut-and-paste” operation along a regular curve of intersection by the same term as its combinatorial equivalent, a *regular exchange*.

If F and F' are compatible, then each curve of intersection between F and F' is a regular curve; therefore, it makes sense to do a regular exchange at each curve of intersection. In this case we obtain a normal surface called the *geometric sum of F and F'* . We shall denote the geometric sum of F and F' by $F + F'$.

There are several interesting properties which are additive with respect to the geometric sum operation. We give two here; another will be given in the next section.

If F and F' are compatible normal surfaces then $F + F'$ is defined and

(1) $\chi(F + F') = \chi(F) + \chi(F')$, where χ is Euler characteristic.

(2) If F corresponds to the n -tuple (x_1, x_2, \dots, x_n) and F' corresponds to the n -tuple (y_1, y_2, \dots, y_n) , then $F + F'$ corresponds to the n -tuple $(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$.

2. LEAST WEIGHT NORMAL SURFACES: EXISTENCE

In this section we provide several theorems showing that whenever a 3-manifold has interesting surfaces, then there exist interesting normal sur-

faces in any triangulation of the manifold. We shall also define our concept of the weight of a normal surface. Together these concepts provide the existence of interesting, least weight normal surfaces.

The major results of this section are well known to 3-manifold topologists and appear in $[K, H_1, H_2]$. A unified approach to several of the existence problems is given in $[J_2]$. However, the relative simplicity and familiarity of these combinatorial techniques are one of the main attractions of our methods over the analytical methods coming from least area (minimal) surface theory. We shall provide an outline of the techniques used in proving existence of interesting, least weight normal surfaces. Since Theorem 2.3 below provides the complete range of techniques for all the existence results needed in this paper, we shall use it as the canonical example.

Let M be a 3-manifold with triangulation \mathcal{T} . We define the *weight of a normal surface F* to be the total number of points in the intersection of F with the 1-skeleton of \mathcal{T} . We use the notation $\text{wt}(F)$ to denote the *weight of F* . This definition can be generalized to give a notion of weight for any (possibly singular) surface missing the 0-skeleton of \mathcal{T} and transverse to the 1-skeleton of \mathcal{T} . The number of vertices of the induced cell structure of a normal surface F is the weight of F . If f_i is the number of i -sided cells in the induced cell structure on F , $i = 3$ or 4 , and $\chi(F)$ is the Euler characteristic of F , then we have:

$$(3) \quad \text{For } F \text{ a closed surface } \text{wt}(F) = \chi(F) + f_4 + \frac{1}{2}f_3;$$

and if F and F' are compatible normal surfaces, then

$$(4) \quad \text{wt}(F + F') = \text{wt}(F) + \text{wt}(F').$$

The following two theorems can be considered as special cases of the work of H. Kneser $[K]$.

THEOREM 2.1. *Let M be a 3-manifold and let \mathcal{H} be a π_1 -invariant subgroup of $\pi_2(M)$. If there is a 2-sphere embedded in M , which is not in \mathcal{H} , then for any triangulation \mathcal{T} of M there is a least weight normal 2-sphere embedded in M , which is not in \mathcal{H} .*

THEOREM 2.2. *Let M be a 3-manifold. If there is a 2-sphere embedded in M , which does not bound a 3-cell in M , then for any triangulation \mathcal{T} of M there is a least weight normal 2-sphere embedded in M , which does not bound a 3-cell in M .*

The next two theorems generalize the work of Kneser referenced above and introduce a new "operation" due to W. Haken $[H_2]$. It was also during this period of time Haken introduced the important properties

of incompressibility and ∂ -incompressibility for surfaces embedded in 3-manifolds.

THEOREM 2.3. *Let M be a 3-manifold, let B be a component of ∂M , and suppose \mathcal{N} is a normal subgroup of $\pi_1(B)$. If there is a disk D embedded in M with ∂D contained in B and $[\partial D]$ not in \mathcal{N} , then for any triangulation \mathcal{T} of M there is a least weight normal disk D' embedded in M with $\partial D'$ contained in B and $[\partial D']$ not in \mathcal{N} .*

Outline of proof. Suppose \mathcal{T} is a triangulation of M .

Step 1. Let D' denote a disk embedded in M such that

- (1) $\partial D'$ is contained in B ,
- (2) $[\partial D']$ is not in \mathcal{N} ,
- (3) D' is in general position with $\mathcal{T}^{(2)}$, the 2-skeleton of \mathcal{T} , and
- (4) D' meets $\mathcal{T}^{(1)}$ in the least number of points of all disks satisfying conditions (1)–(3).

It follows from the hypotheses that we can find such a disk D' .

We now consider the intersection of D' with $\mathcal{T}^{(2)}$.

Step 2. Remove any components of the intersection of D' with $\mathcal{T}^{(2)}$, which are simple closed curves in the interior of a 2-simplex.

This can be done by performing a compression (surgery) on D' beginning at a simple closed curve which is “innermost” on a 2-simplex of \mathcal{T} . After surgery there is always a unique component resulting which is a disk satisfying the same conditions (1)–(3) as D' . We continue to call this disk D' . This step does not introduce any new meets with $\mathcal{T}^{(1)}$ (see Fig. 6).

Step 3. Observe each component of the intersection of D' with a 2-simplex of \mathcal{T} is normally isotopic to a straight spanning arc; i.e., it already is an arc spanning the 2-simplex between different edges.

To see this: Suppose that for some 2-simplex σ of \mathcal{T} there is a component of the intersection between D' and σ , which is an arc having both of

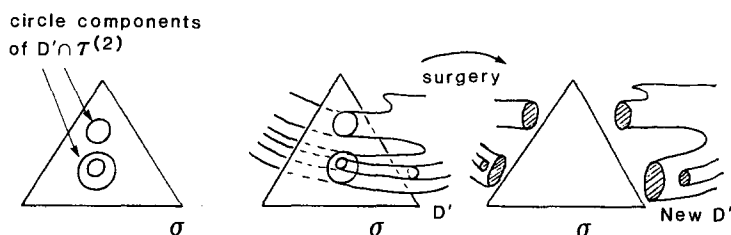


FIGURE 6

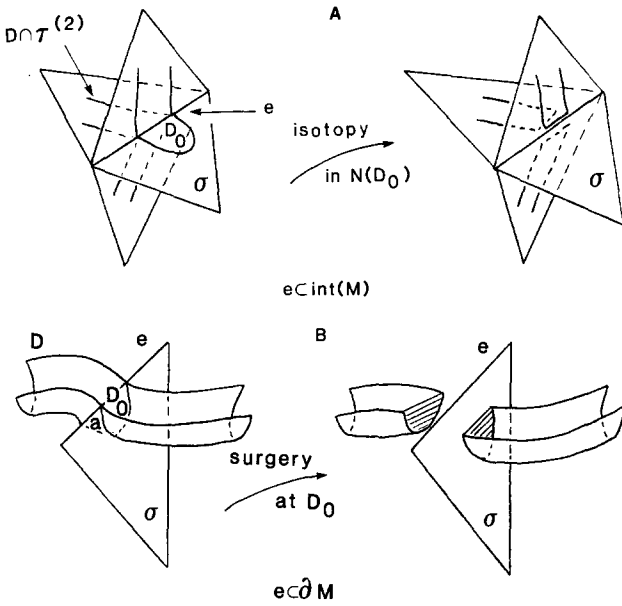


FIGURE 7

its endpoints in the same edge e of σ . There is an “outermost” such arc, a , in σ having both its endpoints in e .

There are two cases to consider depending on e being in the interior of M or e being on the boundary of M .

If such an e were in the interior of M , then an isotopy move of D' would reduce the meets of D' with the 1-skeleton of \mathcal{T} . This is impossible by the choice of D' (see Fig. 7A).

If such an e were on the boundary of M , then it would necessarily be in B and a ∂ -compression (surgery) could be performed. After such a boundary compression there is at least one component resulting which is a disk satisfying the same conditions (1)–(3) as D' , but having fewer intersection points with $\mathcal{T}^{(1)}$. Again, this is impossible by the choice of D' (see Fig. 7B).

Step 4. Observe each component of the intersection of D' with a tetrahedron of \mathcal{T} is a disk.

To see this: Suppose some component of the intersection of D' with a tetrahedron Δ of \mathcal{T} is not a disk. Then there is a simple closed curve component, say α , of the intersection D' with $\partial\Delta$, where the disk P which α bounds in D' is not contained in Δ ; yet, α bounds a disk E in Δ , which does not meet D' except in α . We can replace P by E and obtain a disk satisfying the same conditions (1)–(3) as D' but having fewer intersection points with $\mathcal{T}^{(1)}$. This is impossible by the choice of D' (see Fig. 8).

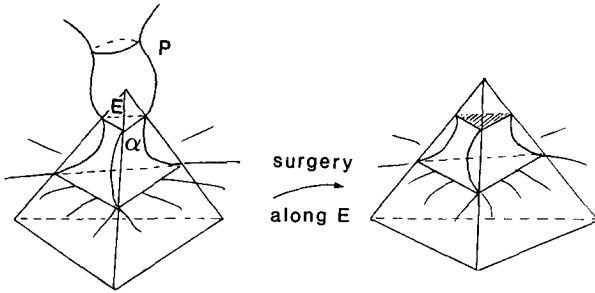


FIGURE 8

Step 5. Observe each component of the intersection of D' with the boundary of a tetrahedron of \mathcal{T} is normally isotopic to a curve type of \mathcal{T} ; i.e., is a simple closed curve which meets a face of the tetrahedron in at most one arc which must span between distinct edges of the tetrahedron.

To see this: Suppose some component of the intersection of D' with the boundary of a tetrahedron on \mathcal{T} is a simple closed curve meeting a face of the tetrahedron in more than one arc. Then there is a component J of the intersection of D' with the boundary of a tetrahedron Δ of \mathcal{T} , where J is an “innermost” simple closed curve on the boundary of Δ and J meets a face σ of Δ in two arcs a and b , where a and b have endpoints in the same edge e of σ and no other points of D' meet e between the endpoints of a and b in e (see Fig. 9).

There are two cases to consider depending on σ being in the interior of M or σ being on the boundary of M .

Now, it follows from Step 4 that J bounds a disk E in Δ (Fig. 9). Since the arcs a and b are in the same face σ of D there is a “fold” in E or

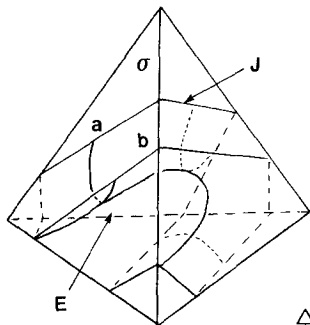


FIGURE 9

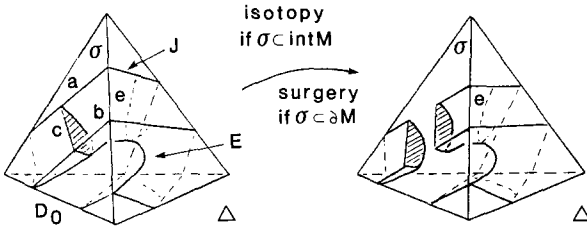


FIGURE 10

equivalently there is a ∂ -compression of E as shown in Fig. 10, where D_0 is a disk and $\partial D_0 = c \cup d$, where $c \subset \sigma$, $d \subset E$, and $D_0 \cap D' = D \cap E = d$.

If σ is in the interior of M , then there is an isotopy of D' creating no new meets with the 1-skeleton of \mathcal{T} and creating a component of intersection with the 2-simplex σ , which is an arc having both its endpoints in the edge e of σ (see Fig. 10). However, by Step 3 above, this situation is impossible.

If σ is in the boundary of M , then necessarily σ is in B and it is possible to perform a boundary compression (surgery) on D' at the disk D_0 . (see Fig. 10). After such a boundary compression there is a component which is a disk satisfying the some conditions (1)–(3) above as D' but having fewer intersections with the 1-skeleton of \mathcal{T} . Again, this is impossible by the choice of D' .

Therefore, having selected D' satisfying conditions (1)–(3) above and meeting the 1-skeleton of \mathcal{T} in the least number of points of all disks satisfying these conditions, then, possibly after the operation as in Step 2, the disk D' is normally isotopic to a normal disk. This is a least weight normal disk having the desired properties. ■

We complete this section by stating a theorem $[H_2]$, which is similar to those above, where the surface now in question is incompressible and ∂ -incompressible. There are several versions of this theorem; however, we shall give the version which requires the hypothesis that the 3-manifold M is irreducible and ∂ -irreducible. Our version is the most used in practice.

THEOREM 2.4. *Let M be an irreducible and ∂ -irreducible 3-manifold. If M contains an embedded, incompressible and ∂ -incompressible surface F , then for any triangulation \mathcal{T} of M there is a least weight normal surface in M isotopic with F .*

The different operations given in Steps 1–5 in the outline of the proof of Theorem 2.3 above provide the necessary techniques to complete the details of a proof for Theorem 2.4. Complete proofs may be found in $[K]$, $[H_2]$, or $[J_2]$.

3. LEAST WEIGHT NORMAL SURFACES: INTERSECTIONS

In this section we investigate the intersection of interesting, least weight normal surfaces. In the analytic theory of least area surfaces, it is at this point the useful properties of least area surfaces are most exploited. For example, if two interesting, embedded surfaces have the property that up to homotopy they need not meet in a 3-manifold, then least area representatives of each of the surfaces either coincide or do not intersect at all. What is surprising is that essentially this same phenomenon is exhibited by interesting, least weight normal surfaces. Of course, least weight normal surfaces have, relative to least area surfaces, freedom of movement via normal isotopy. Hence, intersections can occur in our theory; however, the intersections can be completely understood and are as simple as one might hope.

If two normal surfaces S and S' intersect transversely, the curves of the intersection, $S \cap S'$, which are either simple closed curves or spanning arcs, decompose S and S' into the components of $S - (S \cap S')$ and $S' - (S \cap S')$, respectively. Since each curve of $S \cap S'$ misses the 1-skeleton of \mathcal{T} , we have a well-defined weight for each of these components; and the weight S (weight of S') is the sum of the weights of the components of $S - (S \cap S')$ (of $S' - (S \cap S')$). However, it may be true that some of these components are 0-weight. We will need some observations about such possible 0-weight components.

First, we make a convention about the use of the term *innermost disk*. If S and S' are two embedded surfaces meeting transversely, then a collection of curves of $S \cap S'$ (spanning arcs and simple closed curves) are said to bound an *innermost region* on S , or S' , if they separate off a region of S , or S' , whose interior does not meet $S \cap S'$. We reserve the use of the term *innermost disk* to the special cases when either

- (1) there is a simple closed curve $J \subset S \cap S'$ and J bounds an innermost region that is a disk, or
- (2) there is an arc $a \subset S \cap S'$ and a , along with an arc b in ∂S or $\partial S'$, bounds an innermost region that is a disk.

The point is we may have an innermost region which is a disk but it is not an innermost disk. An innermost disk is special.

When the normal surfaces S and S' meet transversely and E is a cell in the induced cell structure of S or S' , then the curves of $S \cap S'$ meet E in straight spanning arcs. These arcs decompose E into regions and we have possible 0-weight regions in E , which we call 2-edged, 3-edged, or 4-edged, accordingly, as shown in Fig. 11.

A 0-weight component of $S - (S \cap S')$ (or $S' - (S \cap S')$) is given a cell

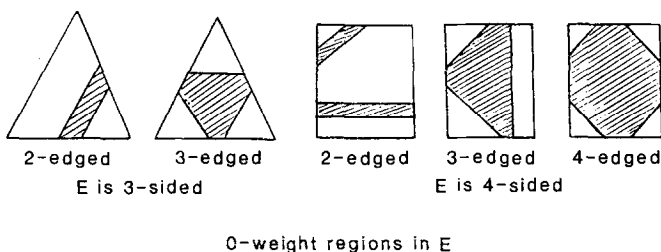


FIGURE 11

structure from the collection of its 0-weight regions in the cells of S (or S'). We are now able to make a crucial observation about 0-weight components.

If X is a 0-weight component of $S - (S \cap S')$ (or $S' - (S \cap S')$), let f_i denote the number of i -edged cells in X , $i = 2, 3, 4$. Let b denote the number of arcs in $\partial X \cap \partial M$, i.e., arcs determined by the intersection of a 0-weight region contained in X and the boundary of the 3-manifold. The Euler characteristic of X , $\chi(X)$, is given by $\chi(X) = -f_4 - \frac{1}{2}f_3 + \frac{1}{2}b$. So, we must have $b \geq 2$ before X can have positive Euler characteristic and be a disk. It follows that

there are no innermost disks of $S - (S \cap S')$ (or $S' - (S \cap S')$)
having 0-weight. (**)

We now consider the intersection between interesting, least weight normal 2-spheres. We shall make precise what we mean by interesting. Let M be a 3-manifold and use \mathcal{H} to denote either

(a) a $\pi_1(M)$ -invariant subgroup of $\pi_2(M)$, or, in a completely distinct consideration,

(b) the collection of embedded 2-spheres in M that bound 3-cells in M . Of course, a 2-sphere will be interesting if it is not in \mathcal{H} . With this notation, we say the 2-sphere S is *equivalent (modulo \mathcal{H})* to the 2-sphere S_1 if in case (a) S and S_1 are in the same coset modulo \mathcal{H} and in case (b) S is isotopic to S_1 .

The next proposition and its corollary provide the necessary analysis of the intersection between interesting, least weight normal 2-spheres.

PROPOSITION 3.1. *Let M be a 3-manifold and let \mathcal{H} be as defined above. Suppose \mathcal{T} is a triangulation of M and S and S' are least weight normal 2-spheres not in \mathcal{H} and meeting transversely in M . Then if $S \cap S'$ is not empty,*

- (1) *there is a curve C of $S \cap S'$, which bounds innermost disks $D \subset S$ and $D' \subset S'$ with $\text{wt}(D) = \text{wt}(D')$,*
- (2) *the 2-sphere $D \cup D'$ is a sphere in \mathcal{H} ,*
- (3) *the curve C of $S \cap S'$ is a regular curve of intersection.*

Proof. Suppose S and S' are least weight normal 2-spheres not in \mathcal{H} and meeting transversely. Let X be an innermost disk in either S or S' , having minimal weight among all innermost disks. Recall from earlier observations, $\text{wt}(X) \neq 0$. The situation is symmetrical: so, let us suppose $X \subset S$ and the boundary of X is the curve $W \subset S \cap S'$.

The curve W separates S into two components with closures X and Y and separates S' into two components with closures, X' and Y' . Furthermore, since X is innermost, $X \cup X'$ and $X \cup Y'$ are both 2-spheres. At least one of the 2-spheres $X \cup X'$ or $X \cup Y'$ is not in \mathcal{H} . Choose notation so that $X \cup Y'$ is not in \mathcal{H} . We now have

$$\text{wt}(X) + \text{wt}(Y') = \text{wt}(X \cup Y') \geq \text{wt}(S') = \text{wt}(X') + \text{wt}(Y');$$

$$\text{and so, } \text{wt}(X) \geq \text{wt}(X').$$

However, by choice of X , we have

$$\text{wt}(X) = \text{wt}(X').$$

It follows that

$$\text{wt}(Y) = \text{wt}(Y').$$

Furthermore, if we consider any point w of $W \cap \mathcal{T}^{(2)}$, then we can make a crucial observation about the intersecting straight arcs in $\mathcal{T}^{(2)}$ determining w . Namely, there is no “fold” at w between the subarcs contained in X and Y' (see Fig. 12). For if there were a “fold” between X and Y' , then there would be an isotopy of $X \cup Y'$ reducing the weight of the 2-sphere $X \cup Y'$. Since $X \cup Y'$ is not in \mathcal{H} , such an isotopy reducing weight would contradict that S and S' realize the least weight of all normal 2-spheres not in \mathcal{H} .

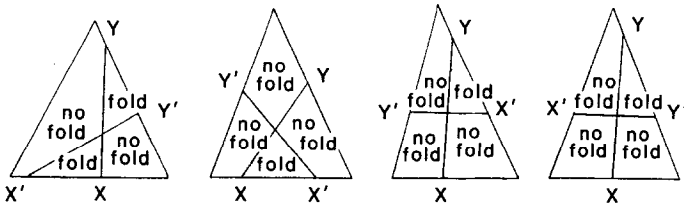


FIGURE 12

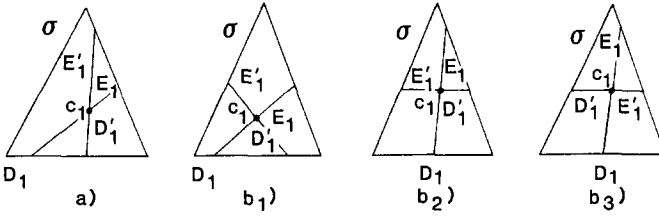


FIGURE 13

The situations presented in Fig. 12 give the various possibilities. It cannot, in general, be concluded that there is a “fold” at any point w along W between X and X' .

Having made these observations, we proceed to find the curve C of $S \cap S'$ satisfying the conclusions of Proposition 3.1.

Let D_1 be an innermost disk in either S or S' having least weight. We assume $D_1 \subset S$. Then D_1 can play the role of X in the previous discussion and we use correspondingly C_1 for W , E_1 for Y , D'_1 for X' , and E'_1 for Y' of that discussion. Then $\text{wt}(D_1) = \text{wt}(D'_1)$, $\text{wt}(E_1) = \text{wt}(E'_1)$, and the 2-sphere $D_1 \cup E'_1$ is not in \mathcal{H} .

Since $\text{wt}(D_1) \neq 0$, there is a 2-simplex σ of \mathcal{T} and a point c_1 of $C_1 \cap \sigma$ having the property that there are no other points of $S \cap S'$ on the subarc in D_1 from c_1 to $\partial\sigma$. See Fig. 13 where we present the four possible situations for the intersection and labeling of the straight arcs of $S \cap \sigma$ and $S' \cap \sigma$ meeting in c_1 .

Notice that we are using the fact that no “fold” can occur between D_1 and E'_1 (D_1 has the role of X and E'_1 has the role of Y' in the above discussion).

Case (a). c_1 is the intersection between arcs in the same normal isotopy class.

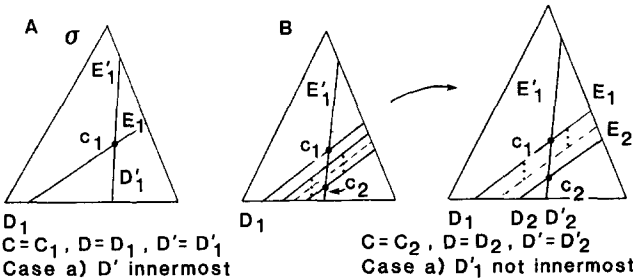


FIGURE 14

In this case there is a “fold” between D_1 and D'_1 . If D'_1 is innermost, (Fig. 14a), then set $C = C_1$, $D = D_1$, and $D' = D'_1$. If D'_1 is not innermost (Fig. 14b), then there must be other arcs of the intersection $S \cap \sigma$ meeting the subarc of $D'_1 \cap \sigma$ between c_1 and $\partial\sigma$.

Let c_2 be the outmost point in this arc, which corresponds to a point of the intersection $S \cap D'_1 \subset S \cap S'$. Let C_2 be the component of $S \cap S'$ containing c_2 (C_2 may very well be C_1).

Then C_2 bounds an innermost disk $D'_2 \subset D'_1 \subset S'$ and the point c_2 of $C_2 \cap \sigma$ has the property that there are no other points of $S \cap S'$ on the surface in D'_2 from c_2 to $\partial\sigma$. Note $\text{wt}(D'_1) = \text{wt}(D'_2)$, so $\text{wt}(D'_2 - D'_1) = 0$. We repeat the previous consideration where the disk D'_2 now plays the role of X and, correspondingly, we use C_2 for W , E'_2 for Y , D_2 for X' , and E_2 for Y' . Of course, $D'_2 \cup E'_2 = S'$ and $D_2 \cup E_2 = S$ and the sphere $D'_2 \cup E_2$ is not in \mathcal{H} .

Since there are no intersections on the subarc in D_2 between c_2 and $\partial\sigma$, both D_2 and D'_2 must be innermost. This is because $\text{wt}(D_2)$ is minimal among disks of $S - S'$, so if there is a curve of $S \cap S'$ in $\text{int}(D_2)$, this curve would have to cross $D_2 \cap \sigma$ between c_2 and $\partial\sigma$. We set $C = C_2$, $D = D_2$, and $D' = D'_2$.

Case (D). c_1 is the intersection between arcs in distinct normal isotopy classes.

In this case there are three, possibly distinct, considerations depending on the labelings. We denote these by (b_1) – (b_3) , respectively, in Fig. 13.

In *Case (b₁)*, there is a “fold” between the subarcs in D_1 and D'_1 and the argument follows the same lines as *Case (a)*.

In *Case (b₂)*, if D'_1 is also innermost, then we can put D'_1 in the role of X as above and, correspondingly, have C_1 for W , E'_1 for Y , E_1 for X' , and D_1 for Y' . Then *Case (b₂)* can be treated as *Case (b₁)*. So, suppose D'_1 is not innermost (See Fig. 15A).

If D'_1 is not innermost, the argument can be carried out exactly as in

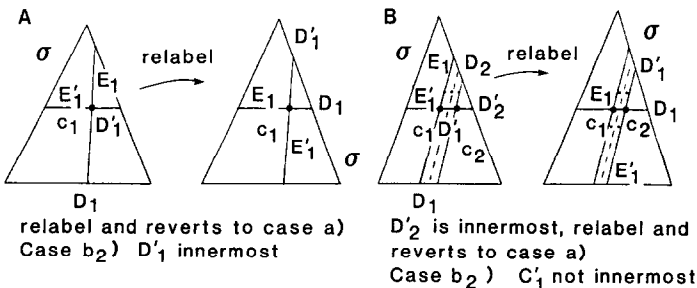


FIGURE 15

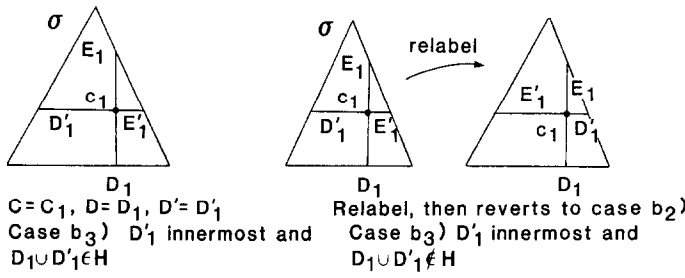


FIGURE 16

Case (a) where the disk denoted D'_1 in Case (a) is not innermost. The corresponding situation is given in Fig. 15b).

In Case (b_3), if D'_1 is also innermost and $D'_1 \cup D_1$ is in \mathcal{H} , then we can set $C = C_1$, $D = D_1$, and $D' = D'_1$. If D'_1 is also innermost and $D'_1 \cup D_1$ is not in \mathcal{H} , then $\text{wt}(D'_1 \cup D_1) \geq \text{wt}(S') = \text{wt}(D'_1 \cup E'_1)$. It follows that $\text{wt}(D_1) = \text{wt}(E'_1)$; but by choice of D_1 , we have the same situation as Case (b) (see Fig. 16).

In Case (b_3) if D'_1 is not innermost, then there must be other arcs of the intersection $S \cap \sigma$ meeting the subarc of $D'_1 \cap \sigma$ between c_1 and $\partial\sigma$ since $\text{wt } D'_1$ is minimal amongst disks of $S' - S$ (see Fig. 17). Let c_2 be the outermost point in this subarc, which corresponds to a point of the intersection $S \cap D'_1 \subset S \cap S'$. Let C_2 be the component of $S \cap S'$ containing c_2 . There are two possible considerations determined by the normal isotopy type of the arc in σ determining c_2 (see Fig. 17).

If c_2 is determined in Case (b_3) by an arc in a normal isotopy class distinct from the class of the arc determining c_1 , then the new labeling is completely determined as in Fig. 18.

However, we now have the same situation as Case (b_1) where there is a "fold" between the subarcs in D_2 and D'_2 and again the argument follows as in Case (a).

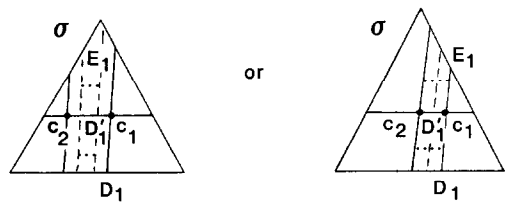
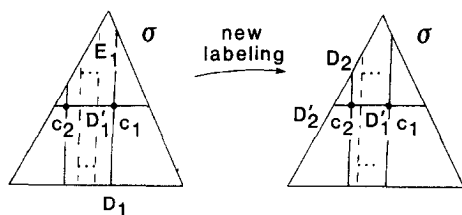
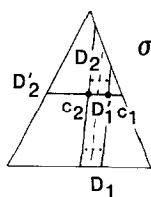


FIGURE 17



Case b_3) Reverts to case a)
using D_2 and D'_2

FIGURE 18



Case b_3) reverts to
case b_2)

FIGURE 19

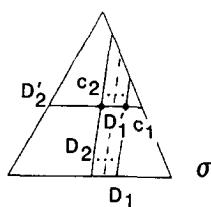
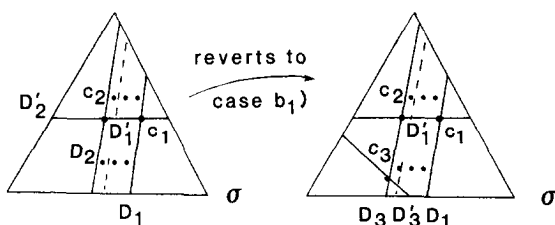


FIGURE 20



Case b_3) D_2 not innermost. Reverts
to case b_1) using D_3, D'_3

FIGURE 21

So, finally suppose c_2 is determined in Case (b₃) by an arc in the same normal isotopy class as the class of the arc determining c_1 . The situation now is dependent upon which subarc determined by c_2 falls into D_2 .

If the subarc in D_2 meets the same edge of σ as E_2 (shown in Fig. 19), then we are in the situation of Case (b₂).

Finally, suppose the subarc in D_2 meets the same edge of σ as D_1 (shown in Fig. 20). If D_2 is also innermost, then we can consider this situation as we did the situation for D'_1 also innermost, at the beginning of Case (b₃); namely, if $D_2 \cup D'_2$ is in \mathcal{H} , then set $C = C_2$, $D = D_2$, and $D' = D'_2$. If $D_2 \cup D'_2$ is not in \mathcal{H} , then change the roles of D_2 and E_2 (i.e., switch labeling) so that the situation is the same as in Case (b₂). If D_2 is not innermost (see Fig. 21), then there must be other arcs of the intersection $S' \cap \sigma$ meeting the subarc of $D_2 \cap \sigma$ between c_2 and $\partial\sigma$. Since D_1 was innermost, the only possibility is as shown in Fig. 21.

Let c_3 be the outermost point in this arc which corresponds to a point of the intersection of $S' \cap D_2 \subset S' \cap S$. Let C_3 be the component of $S' \cap S$ containing c_3 . Then C_3 bounds the innermost disk $D_3 \subset D_2 \subset S$. We can now make an analysis as in Case (b₁).

In all possible situations, we find a curve $C \subset S \cap S'$ with C bounding innermost disks $D \subset S$ and $D' \subset S'$ where $\text{wt}(D) = \text{wt}(D')$; furthermore, for some 2-simplex σ of $\mathcal{T}^{(2)}$ there is a point c of $C \cap \sigma$ and a "fold" between the subarcs of $D \cap \sigma$ and $D' \cap \sigma$ meeting in c . It follows from $\text{wt}(D \cup D') \leq \text{wt}(S) = \text{wt}(S')$ that the 2-sphere $D \cup D'$ is in \mathcal{H} . ■

COROLLARY 3.2. *Let M be a 3-manifold and let \mathcal{H} be as defined above. Suppose \mathcal{T} is a triangulation of M and S and S' are least weight normal 2-spheres not in \mathcal{H} and meeting transversely in M . Then if $S \cap S'$ is not empty*

- (1) *every curve of $S \cap S'$ is a regular curve of intersection,*
- (2) *the geometric sum, $S + S'$, is a disjoint union of two least weight 2-spheres S_1 and S'_1 ($\text{wt}(S_1) = \text{wt}(S)$ and $\text{wt}(S'_1) = \text{wt}(S')$), where S_1 is equivalent to S and S'_1 is equivalent to S' .*

Proof. The proof proceeds via induction on the number of components of the intersection, $S \cap S'$.

If this number is 0, $S \cap S' = \emptyset$, then (1) is vacuous and (2) follows trivially. So, we suppose we know the truth of the corollary for fewer than n components of the intersection, $S \cap S'$, $n \geq 1$.

There is a simple closed curve C , which is a component of $S \cap S'$, satisfying conditions (1)–(3) of the conclusion of Proposition 3.1. Since $D \cup D'$ is a 2-sphere in \mathcal{H} , a regular exchange along C results in an exchange of the disk D in S for the disk D' in S' and vice versa.

Hence, after a regular exchange at C , we can apply the inductive hypothesis. Conclusions (1) and (2) of the corollary follow. ■

There are results similar to those just proved for spheres, which apply to certain least weight normal disks. We state the result for disks without proof.

PROPOSITION 3.3. *Let M be a 3-manifold, let B be a component of ∂M , and let \mathcal{N} be a normal subgroup of $\pi_1(B)$. Suppose \mathcal{T} is a triangulation of M and E and E' are least weight normal disks in M with the property that ∂E and $\partial E'$ are contained in B and $[\partial E]$ and $[\partial E']$ are not in \mathcal{N} . If E and E' meet transversely and $E \cap E'$ is not empty, then*

- (1) *there is a curve C of $E \cap E'$, which bounds innermost disks D and D' on E and E' , respectively, where $\text{wt}(D) = \text{wt}(D')$,*
- (2) *if C is an arc, the boundary of the disk $D \cup D'$ is an element in \mathcal{N} ,*
- (3) *the curve C is a regular curve of intersection.*

Hence,

- (4) *every curve of $E \cap E'$ is a regular curve of intersection,*
- (5) *the geometric sum, $E + E'$, is a disjoint union of two least weight disks E_1 and E'_1 ($\text{wt}(E_1) = \text{wt}(E)$ and $\text{wt}(E'_1) = \text{wt}(E')$) with the property ∂E_1 and $\partial E'_1$ are contained in B and $[\partial E_1]$ and $[\partial E'_1]$ are not in \mathcal{N} .*

We now apply our techniques to the consideration of the intersection between least weight, normal surfaces with positive genus. This requires an additional concept for curves on more general surfaces. A properly embedded arc α in a surface S is *inessential* in S , if α is homotopic ($\text{rel}(\partial\alpha)$) into ∂S . We shall also say a simple closed curve C in a surface S is *inessential* in S , if C is contractible in S . Otherwise, we say such curves are *essential*.

PROPOSITION 3.4. *Let M be an irreducible and ∂ -irreducible 3-manifold. Suppose \mathcal{T} is a triangulation of M and F and F' are incompressible and ∂ -incompressible normal surfaces both of which are least weight in their respective isotopy classes. Furthermore, suppose F and F' intersect transversely. If $F \cap F'$ contains a curve which is inessential in one surface, then it is inessential in the other surface and*

- (1) *there is an inessential curve C of $F \cap F'$ which bounds innermost disks $D \subset F$ and $D' \subset F'$ with $\text{wt}(D) = \text{wt}(D')$,*
- (2) *if C is an arc, the disk $D \cup D'$ is parallel to a disk in ∂M and if C is a simple closed curve, the sphere $D \cup D'$ bounds a 3-cell in M ,*
- (3) *the curve C is a regular curve of intersection.*

Hence,

- (4) every inessential curve of $F \cap F'$ is a regular curve of intersection,
- (5) the result of regular exchanges at each inessential curve of $F \cap F'$ is a union of two normal surfaces F_1 and F'_1 , which are least weight in their respective isotopy classes, where F_1 is isotopic to F and F'_1 is isotopic to F' and no curve of $F_1 \cap F'_1$ is inessential.

Proof. If there is a curve in $F \cap F'$ which is inessential in one of the surfaces, then it is inessential in the other surface. So, the consideration of inessential curves in $F \cap F'$ is symmetrical in F and F' . Now, assume there is an inessential curve in $F \cap F'$. Then there are inessential curves in $F \cap F'$ which bound innermost disks, some on F and some on F' . Consider all innermost disks and suppose X is one having least weight. Again from the observations earlier in this section $\text{wt}(X) \neq 0$. Let W be the curve of $F \cap F'$ determining X . We may assume notation has been chosen so that $X \subset F$.

The curve W separates F into two components with closures X and Y and separates F' into two components with closures X' and Y' , where X' , say, is also a disk. The surface $X \cup Y'$ is isotopic to F' and so $\text{wt}(X \cup Y') \geq \text{wt}(F') = \text{wt}(X' \cup Y')$. It follows that $\text{wt}(X) = \text{wt}(X')$ and so, $\text{wt}(Y) = \text{wt}(Y')$.

Since neither F nor F' are assumed to be a disk or a 2-sphere, we have that neither Y nor Y' are disks. Therefore, if both X and X' are innermost, then the surfaces $X \cup Y'$ and $X' \cup Y$ are isotopic to F' and F , respectively, and $\text{wt}(X \cup Y') = \text{wt}(F')$, $\text{wt}(X' \cup Y) = \text{wt}(F)$. It follows that for any point of $W \cap \mathcal{T}^{(2)}$ there are no "folds" between X and Y' , as well as, no "folds" between X' and Y ; so, the curve W is regular. See Fig. 2 to verify this. The fold "changes position" along the exceptional arc of intersection of different 4-sided disk types. (During the considerations in the proof of Proposition 3.1, when the surfaces had genus 0, it was impossible to conclude that the curve W was regular at a similar point in the argument. In that argument we needed to have the 2-sphere $D \cup D'$ in \mathcal{H} . Knowing W is regular is a great advantage. While we could make an argument, investigating the various possibilities as in the proof of Proposition 3.1, we shall not. Knowing the curve W is regular, we can give a much more elegant argument, which has merit of its own.)

Suppose there are inessential simple closed curve components of $F \cap F'$. Let X_0 be an innermost disk bounded by a simple closed curve W_0 in $F \cap F'$ with least weight among all such disk. Say $X_0 \subset F$. Then there is a disk $X'_0 \subset F'$ with $W_0 = \partial X'_0$ and the 2-sphere $X_0 \cup X'_0$ bounds a 3-cell. If X'_0 is also innermost, then set $D = X_0$, $D' = X'_0$, and $C = W_0$.

If X'_0 is not innermost, then there is a simple closed curve $W_1 \subset X'_0 \cap F$ and an innermost disk $X'_1 \subset X'_0$ with $W_1 = \partial X'_1$. Now, we have a disk

$X_1 \subset F$ with $W_1 = \partial X_1$ and the 2-sphere $X_1 \cup X'_1$ bounds a 3-cell. Note, X_0 may be contained in X_1 .

In this fashion we inductively define sequences of distinct, least weight disks X_0, \dots, X_n in F , X'_0, \dots, X'_n in F' , and a sequence of simple closed curves W_0, \dots, W_n , where $W_i = \partial X_i = \partial X'_i$, having the properties: $X_i \not\subset X_j$, $X'_i \not\subset X'_j$, $0 \leq j < i \leq n$, X_0, X_2, \dots , are innermost, and X'_1, X'_3, \dots , are innermost.

If X_n and X'_n are both innermost, set $D = X_n$, $D' = X'_n$, and $C = W_n$.

Now, by construction, either X_n or X'_n is innermost. The argument is symmetrical; so, let's say X'_n is innermost and X_n is not innermost. Then there is a simple closed curve $W_{n+1} \subset X_n \cap F'$ and an innermost disk $X_{n+1} \subset X_n$ with $W_{n+1} = \partial X_{n+1}$. If $X_i \subset X_n$ is innermost, then choose $W_{n+1} = W_i$ and $X_{n+1} = X_i$.

We have a disk $X'_{n+1} \subset F'$ with $W_{n+1} = \partial X'_{n+1}$ and the 2-sphere $X_{n+1} \cup X'_{n+1}$ bounds a 3-cell. Of course, if $X_{n+1} = X_i$, then $X'_{n+1} = X'_i$.

Eventually, we have either X_n and X'_n both innermost, or for some i , $0 \leq i \leq n$, the innermost disk $X_i \subset X_n$ (or innermost $X'_i \subset X'_n$). If X_n and X'_n are both innermost, then we have found the desired disk pair. So, suppose (possibly changing indices if necessary) for n , we have $X_0 \subset X_n$ and this is the first such occurrence (see Fig. 22).

In this situation, replace $X'_i \subset F'$ by X_i , if X_i is innermost, and replace $X_i \subset F$ by X'_i , if X'_i is innermost. We obtain new surfaces F_1 isotopic to F and F'_1 isotopic to F' with $\text{wt}(F_1) = \text{wt}(F)$ and $\text{wt}(F'_1) = \text{wt}(F')$. However, there is a torus $T = (X'_0 - X'_1) \cup (X_1 - X_2) \cup \dots \cup (X_n - X_0)$, which is a normal torus (since each W_i is a regular curve) and $\text{wt}(T) = \text{wt}(X'_0 - X'_1) + \dots + \text{wt}(X_n - X_0)$ cannot be zero. Since $\text{wt}(F_1) + \text{wt}(F'_1) \geq \text{wt}(F) + \text{wt}(F') = \text{wt}(F_1) + \text{wt}(F'_1) + \text{wt}(T)$, we have a contradiction. We conclude that for some n both X_n and X'_n must be innermost.

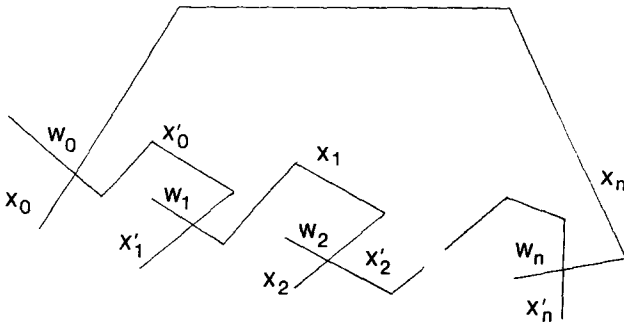


FIGURE 22

The argument in the case when all inessential curves are spanning arcs follows along the same lines with the single exception that the surface T is a normal annulus as opposed to a normal torus.

We now have satisfied conditions (1)–(3). The remainder of the argument follows exactly as in the proof of Corollary 3.2. ■

Suppose F_0 and F_1 are embedded surfaces in the 3-manifold M and intersect transversely in M . The subsurfaces $S_0 \subset F_0$ and $S_1 \subset F_1$ are *parallel* (in M) if there is a submanifold R of M where $S_0 \cup S_1 = \partial R$, $S_0 \cap S_1 = \partial S_0 = \partial S_1$, and the pair (R, S_i) is homeomorphic to the pair $(S_0 \times [0, 1], S_0 \times i)$, $i = 0, 1$. We allow the surfaces F_0 and F_1 to meet the interior of R and refer to R as a *region of parallelity* between S_0 and S_1 .

PROPOSITION 3.5. *Let M be an irreducible and ∂ -irreducible 3-manifold. Suppose \mathcal{T} is a triangulation of M and F_0 and F_1 are incompressible and ∂ -incompressible normal surfaces in M , both of which are least weight in their respective isotopy classes. If F_0 and F_1 intersect transversely, then for any pair of subsurfaces $S_0 \subset F_0$ and $S_1 \subset F_1$, which are innermost, as well as parallel, in M ,*

$$(1) \quad \text{wt}(S_0) = \text{wt}(S_1),$$

$$(2) \quad \text{the curves } \partial S_0 = \partial S_1 \text{ are regular curves of intersection,}$$

$$(3) \quad \text{a regular exchange at each curve of } \partial S_0 = \partial S_1 \text{ results in two normal surfaces } F'_0 \text{ and } F'_1 \text{ both of which are least weight in their respective isotopy classes, where } F'_i \text{ is isotopic to } F_i, i = 0, 1.$$

Proof. Suppose the subsurfaces $S_0 \subset F_0$ and $S_1 \subset F_1$ are innermost, as well as parallel, in M . The surfaces $F'_0 = (F_0 - S_0) \cup S_1$ and $F'_1 = (F_1 - S_1) \cup S_0$ are isotopic to F_0 and F_1 , respectively. Therefore, $\text{wt}(F_0) \leq \text{wt}(F'_0)$ and $\text{wt}(F_1) \leq \text{wt}(F'_1)$. By the least weight assumptions on F_0 and F_1 , we have $\text{wt}(S_1) \geq \text{wt}(S_0)$ and $\text{wt}(S_0) \geq \text{wt}(S_1)$. It follows $\text{wt}(S_0) = \text{wt}(S_1)$.

Now, $\text{wt}(F_i) = \text{wt}(F'_i)$, $i = 0, 1$ implies there are no “folds” along any of the curves of $\partial S_0 = \partial S_1$ between the pieces of $(F_0 - S_0)$ and S_1 or between the pieces of $(F_1 - S_1)$ and S_0 . This forces all the curves of $\partial S_0 = \partial S_1$ to be regular curves. (See Fig. 2.)

Thus we have established Claims (1) and (2). Claim (3) follows immediately. ■

COROLLARY 3.6. *In addition to the hypotheses of Proposition 3.5 assume no component of $F_0 \cap F_1$ is an inessential curve. If $S_0 \subset F_0$ and $S_1 \subset F_1$ are subsurfaces (not necessarily innermost) which are parallel in M , then each curve of $\partial S_0 = \partial S_1$ is a regular curve of intersection.*

Proof. There are no inessential curves of $F_0 \cap F_1$; so, if there are sub-surfaces of F_0 and of F_1 , which are parallel in M , then there are subsurfaces T_0 of F_0 and T_1 of F_1 which are innermost, as well as parallel, in M (cf. [Wa]). Proposition 3.5 establishes that each curve of $\partial T_0 = \partial T_1$ is a regular curve of intersection. If $S_i = T_i$, $i = 0, 1$, then we are done. So, suppose this is not the case. A regular exchange at each curve of $\partial T_0 = \partial T_1$ results in two normal surfaces F'_0 and F'_1 , both of which are least weight in their respective isotopy classes (F'_i is isotopic with F_i , $i = 0, 1$). Furthermore, there are subsurfaces S'_i of F'_i , which are parallel in M , and $\partial S'_i = \partial S_i$, $i = 0, 1$.

There are fewer curves of $F'_0 \cap F'_1$ than of $F_0 \cap F_1$; so, if we argue by induction we can conclude that each curve of $\partial S'_0 = \partial S'_1$ is a regular curve. Since $\partial S'_i = \partial S_i$, $i = 0, 1$, we have established Corollary 3.6. ■

If F_0 and F_1 are properly embedded surfaces in the 3-manifold M , we say they are *homotopically disjoint* if there exist surfaces F'_0 and F'_1 such that $F'_0 \cap F'_1 = \emptyset$, where F_0 is homotopic to F'_0 and F_1 is homotopic to F'_1 .

The following proposition is an immediate consequence of the previous material of this section and standard arguments from [Wa].

PROPOSITION 3.7. *Let M be an irreducible and ∂ -irreducible 3-manifold. Suppose \mathcal{T} is a triangulation of M and F_0 and F_1 are incompressible and ∂ -incompressible normal surfaces, both of which are least weight in their respective isotopy classes. Furthermore, suppose F_0 and F_1 intersect transversely and are homotopically disjoint. Then*

- (1) *each curve of $F_0 \cap F_1$ is a regular curve of intersection,*
- (2) *the geometric sum $F_0 + F_1$ is a disjoint union of two normal surfaces F'_0 and F'_1 , both of which are least weight in their respective isotopy classes, where F'_0 is isotopic to F_0 and F'_1 is isotopic to F_1 .*

4. EQUIVARIANT SPHERE THEOREM/LOOP THEOREM

In this section we use least weight normal surfaces to perform equivariant operations with respect to certain group actions on 3-manifolds. These methods provide a PL environment in which to obtain results analogous to those obtained, using least area surfaces, in the work of Meeks and Yau and of Meeks, Simon, and Yau [M-Y₂, M-Y₃, M-S-Y].

Throughout, we will have a fixed triangulation of the 3-manifold M and consider groups of simplicial homeomorphisms. Of course, our results can be immediately applied to smooth 3-manifolds and properly discontinuous groups of diffeomorphisms. (We define a group G of homeomorphisms to be *properly discontinuous* if every point of M has an open neighborhood U

which meets only finitely many of its translates under G . Note that any group of simplicial homeomorphisms is properly discontinuous.) If X is a subset of M and G is a properly discontinuous group of homeomorphisms of M , the set X is G -equivariant if for each $g \in G$ either $g(X) = X$ or $g(X) \cap X = \emptyset$. The set X is G -invariant if for each $g \in G$, $g(X) = X$. The set $\text{Fix}(G) \equiv \{p \in M : g(p) = p \text{ for some } g \in G\}$ is the *fixed point set* of G .

The first theorem in this section (Theorem 4.1) is the equivariant version of the Sphere Theorem. We also state without proof the equivariant version of the Loop Theorem (Theorem 4.3). Our techniques enable us to give identical proofs of these two results. We use the classical Loop Theorem $[P_2, St_1]$ in the proofs of both results; however, we do not need to employ the so-called "tower construction" in our argument. The classical version of the Sphere Theorem $[P_1, W, E]$ is an immediate corollary of Theorem 4.1. Our proof restricted to this particular case provides a combinatorial proof of the Sphere Theorem using the Loop Theorem and avoiding the "tower-construction."

THEOREM 4.1 (Equivariant Sphere Theorem). *Let M be a 3-manifold with triangulation \mathcal{T} and suppose G is a group of simplicial homeomorphisms of M with $\text{Fix}(G)$ a subcomplex. Let \mathcal{H} be a $\pi_1(M)$ and G -invariant proper subgroup of $\pi_2(M)$. Then there is an embedded, G -equivariant 2-sphere or two-sided projective plane S in M , whose class does not represent an element in \mathcal{H} .*

Proof. Let (\tilde{M}, p) be the universal covering of M , where $p: \tilde{M} \rightarrow M$ is the covering projection. Then \tilde{M} has a triangulation \mathcal{T} lifted from the triangulation \mathcal{T} of M . Furthermore, the collection of all lifts of elements of G into the group of homeomorphisms of \tilde{M} generates a group of simplicial homeomorphisms \tilde{G} of \tilde{M} with $\text{Fix}(\tilde{G})$ a subcomplex. There is an exact sequence $1 \rightarrow \pi_1(M) \rightarrow \tilde{G} \rightarrow G \rightarrow 1$, where $\pi_1(M)$ represents the normal subgroup of \tilde{G} consisting of the covering transformations of (\tilde{M}, p) . The subgroup $\tilde{\mathcal{H}} = p_*^{-1}(\mathcal{H})$ is a proper \tilde{G} -invariant subgroup of $\pi_2(\tilde{M})$.

Claim. There is an embedded 2-sphere \tilde{S} in \tilde{M} whose class does not represent an element in $\tilde{\mathcal{H}}$. (In other words, the Sphere Theorem has a solution in the simply connected 3-manifold \tilde{M} .)

Proof. Since \tilde{M} is simply connected, $H_2(\tilde{M})$ is isomorphic with $\pi_2(\tilde{M})$. Furthermore, any class in $H_2(\tilde{M})$ is represented by a closed, embedded, orientable 2-manifold. Since $\tilde{\mathcal{H}}$ is a proper subgroup of $\pi_2(\tilde{M}) (\approx H_2(\tilde{M}))$, there is a closed, embedded, orientable 2-manifold \tilde{F} whose class does not represent an element in $\tilde{\mathcal{H}}$ (we can assume \tilde{F} connected). Now suppose \tilde{F} also has smallest genus of all such embedded, connected surfaces representing classes not in $\tilde{\mathcal{H}}$.

If \tilde{F} is not a 2-sphere, then by the Loop Theorem there is a disk D in \tilde{M}

with $D \cap \tilde{F} = \partial D$, a noncontractible loop in \tilde{F} . We may perform surgery on \tilde{F} along D , to obtain a new representative of a class not in \mathcal{H} but with smaller genus than the genus of \tilde{F} . This contradicts the choice of \tilde{F} ; so, \tilde{F} is a 2-sphere.

This establishes the claim.

By Theorem 2.1, there is a least weight normal (with respect to $\tilde{\mathcal{T}}$) 2-sphere in \tilde{M} which is not in \mathcal{H} . We shall denote such a 2-sphere by \tilde{S} .

Let us first consider the situation when the group G (and hence \tilde{G}) is acting freely ($\text{Fix}(\tilde{G}) = \emptyset$). Since \tilde{G} is a group of simplicial homeomorphisms, for each $g \in \tilde{G}$, $g(\tilde{S})$ is a least weight normal 2-sphere not representing a class in \mathcal{H} . We may assume for $g \in \tilde{G}$, either g leaves \tilde{S} invariant ($g(\tilde{S}) = \tilde{S}$), or $g(\tilde{S})$ meets \tilde{S} transversely, or $g(\tilde{S}) \cap \tilde{S} = \emptyset$, by making \tilde{S} transverse to the projection from \tilde{M} to \tilde{M}/\tilde{G} .

We define a complexity for a normal surface with respect to the action of the group \tilde{G} . If X is a normal surface (with respect to $\tilde{\mathcal{T}}$), let $X^{(1)}$ denote the 1-skeleton of the induced cellular structure of X . If $g \in \tilde{G}$, we define the *complexity between X and gX* , written $\mathcal{C}(X, gX)$, to be the number of points in $X^{(1)} \cap gX^{(1)}$, if X and gX meet transversely, and 0, if $X = gX$ or $X \cap gX = \emptyset$. The *complexity of X* (with respect to the action of the group \tilde{G}) is $\mathcal{C}(X, \tilde{G}) = \sum_{g \in \tilde{G}} \mathcal{C}(X, gX)$. We shall use $\mathcal{C}(X)$ when the group \tilde{G} is understood. *The complexity $\mathcal{C}(X) = 0$ if and only if X is \tilde{G} -equivariant.*

Now, among the nonempty collection of least weight, normal 2-spheres not representing a class in \mathcal{H} , let \tilde{S} be one having $\mathcal{C}(\tilde{S})$ a minimum. We shall prove \tilde{S} is \tilde{G} -equivariant.

If \tilde{S} is not \tilde{G} -equivariant, then for some $\tau \in \tilde{G}$, $\tau(\tilde{S})$ meets \tilde{S} transversely in a nonempty collection of simple closed curves. Notice that the complexity between \tilde{S} and $\tau\tilde{S}$ is the number of points of $\tilde{S} \cap \tau\tilde{S} \cap \tilde{\mathcal{T}}^{(2)}$, which is just the number of times the curves $\tilde{S} \cap \tau\tilde{S}$ meet the 1-skeleton of the induced cellular structures of \tilde{S} or $\tau\tilde{S}$.

We shall first show we may assume τ has finite order. For suppose τ has infinite order and n is the largest positive integer so that $\tilde{S} \cap \tau^n \tilde{S} \neq \emptyset$. The two 2-spheres \tilde{S} and $\tau^n \tilde{S}$ satisfy the hypothesis of Lemma 3.1. So, there is a curve C of $\tilde{S} \cap \tau^n \tilde{S}$, which bounds innermost disks $D \subset \tilde{S}$ and $D' \subset \tau^n \tilde{S}$, $\text{wt}(D) = \text{wt}(D')$, and the 2-sphere $D \cup D'$ is a sphere in \mathcal{H} . Furthermore, the curve C of $\tilde{S} \cap \tau^n \tilde{S}$ is a regular curve of intersection. We now consider a 2-sphere S^* obtained by regular exchanges at C and/or $\tau^{-n}C$. For these exchanges to make sense, we need to have $C \cap \tau^{-n}C \neq \emptyset$. This is precisely our reason for the choice of n ; because if $C \cap \tau^{-n}C \neq \emptyset$, then $\tau^{-n}\tilde{S} \cap \tilde{S} \cap \tau^n \tilde{S} \neq \emptyset$, which implies $\tau^{2n}\tilde{S} \cap \tilde{S} \neq \emptyset$. This contradicts n being the largest positive integer so that $\tau^n \tilde{S} \cap \tilde{S} \neq \emptyset$.

The precise construction of S^* depends on the possible ways $\tau^{-n}D'$ can be placed in \tilde{S} (see Fig. 23). The possibilities are $\tau^{-n}D' \cap D = \emptyset$ (Fig. 23a), $\tau^{-n}D' \subset D$ (Fig. 23b), and $\tau^{-n}D' \supset D$ (Fig. 23c).

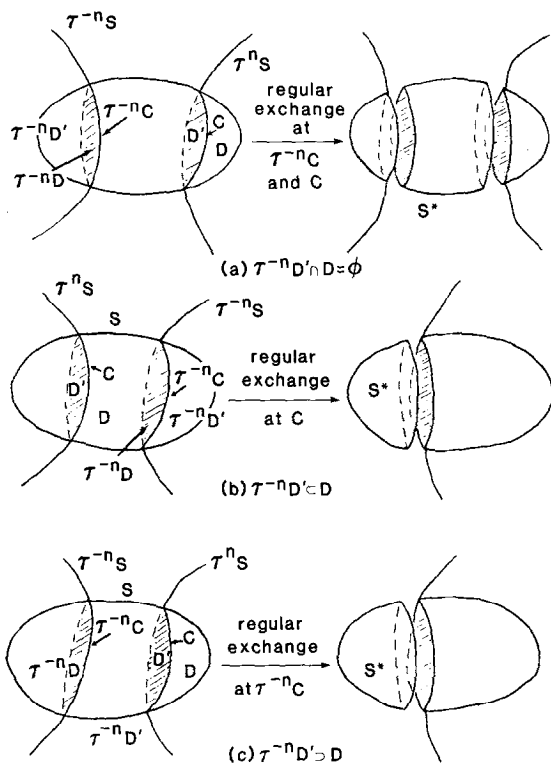


FIGURE 23

If $\tau^{-n}D' \cap D = \emptyset$, then S^* is obtained by making regular exchanges at both C and $\tau^{-n}C$ (see Fig. 23a). Since C ($\tau^{-n}C$) is innermost on both \tilde{S} and $\tau^{-n}\tilde{S}$ (\tilde{S} and $\tau^{-n}\tilde{S}$), S^* is an embedded 2-sphere. Since $D \cup D'$ and $\tau^{-n}D \cup \tau^{-n}D'$ are both in \mathcal{H} , S^* is not in \mathcal{H} . Since C and $\tau^{-n}C$ are regular, S^* is normal. Since $\text{wt}(D) = \text{wt}(D')$ and $\text{wt}(\tau^{-n}D) = \text{wt}(\tau^{-n}D')$, $\text{wt}(S^*) = \text{wt}(\tilde{S})$.

If $\tau^{-n}D' \subset D$, then S^* is obtained by a regular exchange at C (see Fig. 23b). Since C is innermost on $\tau^n\tilde{S}$, S^* is an embedded 2-sphere. Since $D \cup D'$ is in \mathcal{H} , S^* is not in \mathcal{H} . Since C is regular, S^* is normal. Since $\text{wt}(D) = \text{wt}(D')$, $\text{wt}(S^*) = \text{wt}(\tilde{S})$.

If $\tau^{-n}D' \supset D$, then by exchanging the roles of τ^n and τ^{-n} , we have the same situation as that just considered (we make a regular exchange at $\tau^{-n}C$ (see Fig. 23c).

In all cases the 2-sphere S^* is an embedded, least weight normal 2-sphere. We claim $\mathcal{C}(S^*) < \mathcal{C}(\tilde{S})$.

The 2-sphere S^* (or equivalently, the 1-skeleton of the induced cellular

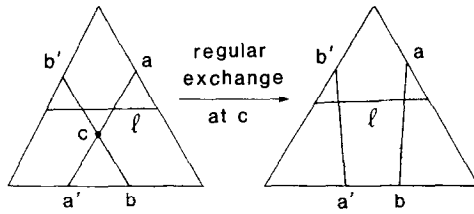


FIGURE 24

structure of S^*) is determined by regular exchanges at the points in the 1-skeleton of \tilde{S} , where the curve C and/or the curve $\tau^{-n}C$ cross the 1-skeleton of \tilde{S} . The orbit of the 1-skeleton of S^* is then determined by regular exchanges in the orbits of these points, none of which ever show up in the 1-skeleton of \tilde{S} except along C or $\tau^{-n}C$. To consider the complexity of S^* it is sufficient to consider the effects of these regular exchanges one at a time. Of course, we only need to consider the finite number of such regular exchanges which take place in that part of the 2-skeleton of $\tilde{\mathcal{T}}$ which contains the 1-skeleton of \tilde{S} .

So, suppose c is a point where there is to be a regular exchange. Then c is an intersection point of two straight line segments aca' and ccb' in the 2-simplex σ of $\tilde{\mathcal{T}}$ (see Fig. 24).

If c is a point of C or $\tau^{-n}C$, we can choose labeling so that the segment ac is in $S - (D \cup \tau^{-n}D')$ and the segment cb is in D' (c in C) or in $\tau^{-n}D$ (c in $\tau^{-n}C$). If c is a point in the orbit of C or the orbit of $\tau^{-n}C$, we can choose labeling so that the segment ac is in the orbit of $S - (D \cup \tau^{-n}D')$ and the segment cb is in the orbit of D' (c in the orbit of C) or the orbit of $\tau^{-n}D$ (c in the orbit of $\tau^{-n}C$). After the regular exchange, the line segment ab is in S^* (c in C or $\tau^{-n}C$) or in the orbit of S^* (c in the orbit of C or the orbit of $\tau^{-n}C$). Now, we observe: *a line segment l in σ meets the line segment ab if and only if l meets either the line segment ac or the line segment cb , but does not meet both.*

We consider a regular exchange at c as in Fig. 24 in two stages: first transform $b'cb$ and aca' into $b'ca'$ and acb , then straighten these lines into $b'a'$ and ab .

If $\tau^{-n}D' \cap D = \emptyset$, we begin with a complexity equal to the number of intersection points over the 1-skeleton of $\tilde{S} - (D \cup \tau^{-n}D')$ plus the number of intersection points over the 1-skeleton of D' plus the number of intersection points over the 1-skeleton of $\tau^{-n}D$ less the number of times C and $\tau^{-n}C$ meet the 1-skeleton of \tilde{S} ; i.e., the complexity of S less the complexity along C and $\tau^{-n}C$, where we make regular exchanges. If a regular exchange is made at a point c of C or $\tau^{-n}C$, then by the above observation we do not increase complexity. Similarly for regular exchanges at points c in the

orbit of C or the orbit of $\tau^{-n}C$. So, in this case $\mathcal{C}(S^*) < \mathcal{C}(\tilde{S})$. This is a contradiction to the choice of \tilde{S} .

If $\tau^{-n}D' \subset D$, we begin with a complexity equal to the number of intersection points over the 1-skeleton of $(\tilde{S} - D)$ plus the number of intersection points over the 1-skeleton of D' less the number of times C meets the 1-skeleton of \tilde{S} . A priori this number seems like it could be large, depending on the number of intersection points in the 1-skeleton of D' ; however, $\tau^{-n}D' \subset D$ implies this number is no larger than the number in the 1-skeleton of D . It follows that we start with a number of intersection points no larger than the complexity of \tilde{S} less the complexity along C . Again the regular exchanges at points c of C or in the orbit of C cannot increase this complexity. We conclude in this case $\mathcal{C}(S^*) < \mathcal{C}(\tilde{S})$. This is a contradiction to the choice of \tilde{S} .

As we observed before, the case $\tau^{-n}D' \supset D$ is just the preceding case with the roles of τ^{-n} and τ^n reversed.

We have shown that τ having infinite order leads to a contradiction of our choice of \tilde{S} . So, we may suppose τ has finite order and $\tau\tilde{S} \cap \tilde{S} \neq \emptyset$, where $\tau\tilde{S} \neq \tilde{S}$.

Assume $n > 0$ is the order of τ . Consider the collection of 2-spheres $\tilde{S}, \tau(\tilde{S}), \dots, \tau^{k-1}(\tilde{S})$, where $\tau^k(\tilde{S}) = \tilde{S}$, $0 < k \leq n$. If $gp\langle\tau\rangle$ is the (finite cyclic) group generated by τ , then the set $\tilde{S} \cup \tau\tilde{S} \cup \dots \cup \tau^{k-1}\tilde{S}$ is $gp\langle\tau\rangle$ -invariant. By Corollary 3.2, this collection of 2-spheres is a compatible collection and it makes sense to consider the geometric sum

$$\tilde{S} + \tau\tilde{S} + \dots + \tau^{k-1}\tilde{S}. \quad (****)$$

(We digress for a moment to consider this geometric sum. The geometric sum between two compatible normal surfaces is formed by making regular exchanges along the curves of intersection (all of which are regular curves). But when more than two normal surfaces are involved, there are triple points and there is naturally a question of the order we do the exchanges. (Is geometric addition associative?) Or, after the addition of two surfaces, are the resulting normal surfaces still compatible? One of the beauties of normal surface theory is that all of these operations can be considered merely by performing regular exchanges in the 2-skeleton of the triangulation \mathcal{T} at points where the 1-skeleta of the surfaces intersect. Hence, geometric addition is associative. Furthermore, the geometric addition of two normal surfaces does not introduce curve types other than those already present; hence, compatibility of geometric addition between more than two compatible surfaces does not become a problem. These facts also follow from the observation in Section 2 that the correspondence from normal isotopy classes of normal surfaces to tuples of nonnegative integers takes geometric addition to standard linear addition in Euclidean space.)

The geometric sum (****) is a pairwise disjoint collection of normal 2-spheres $\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_{k-1}$, where $\text{wt}(\tilde{S}_i) = \text{wt}(\tilde{S})$, $0 \leq i < k$, \tilde{S}_i is equivalent to $\tau^i \tilde{S}$ (modulo \mathcal{H}), and the collection is invariant under $gp\langle \tau \rangle$. Since \tilde{S} is not in \mathcal{H} and \mathcal{H} is G -invariant, it follows that \tilde{S}_i is not in \mathcal{H} , $0 \leq i < k$.

We claim some \tilde{S}_i has $\mathcal{C}(\tilde{S}_i) < \mathcal{C}(\tilde{S})$.

We begin with the collection of 1-skeleta for the 2-spheres $\tilde{S}, \tau\tilde{S}, \dots, \tau^{k-1}\tilde{S}$. There is a complexity associated with this union of 1-skeleta and it is precisely $k \cdot \mathcal{C}(S)$. The 1-skeleta for the 2-spheres $\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_{k-1}$ are the results of regular exchanges at the points where the 1-skeleton of $\tau^i \tilde{S}$ meets the 1-skeleton of $\tau^j \tilde{S}$ for all pairs i, j with $0 \leq i < j \leq k-1$. The orbits of the 1-skeleta of $\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_{k-1}$ are the results of regular exchanges at the points where the 1-skeleton $g\tau^i \tilde{S}$ meets the 1-skeleton $g\tau^j \tilde{S}$ taken over all pairs i, j with $0 \leq i < j \leq k-1$, and all $g \in G$. (Notice the only regular exchanges occurring in the 1-skeleta of $\tilde{S}, \tau\tilde{S}, \dots, \tau^{k-1}\tilde{S}$ are at the points where the 1-skeleton of some $\tau^i \tilde{S}$ meets the 1-skeleton $\tau^j \tilde{S}$, $i \neq j$.) The 1-skeleta of $\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_{k-1}$ are carried by a finite number of 2-simplices of \mathcal{T} (namely, the 2-simplices of \mathcal{T} , which carry the 1-skeleton of $\tilde{S}, \tau\tilde{S}, \dots, \tau^{k-1}\tilde{S}$). The regular exchanges at points of intersection of the 1-skeleton of $\tau^i \tilde{S}$ with $\tau^j \tilde{S}$, $0 \leq j < j \leq k-i$, all occur in these same 2-simplices. Furthermore, only the points in the orbits of these points of intersection, which are carried by this same collection of 2-simplices, affect the complexities of the spheres $\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_{k-1}$. So, only a finite number of regular exchanges need to be considered to calculate (bound) the complexities of the spheres $\tilde{S}_0, \dots, \tilde{S}_{k-1}$.

If a regular exchange is made at a point c of intersection of the 1-skeleton of $\tau^i \tilde{S}$ with the 1-skeleton of $\tau^j \tilde{S}$, $i \neq j$, then we eliminate the intersection point c and, by our earlier observation, only reduce other possible intersections (see Fig. 24). If a regular exchange is made at a point c in the G -orbit of a point of intersection of the 1-skeleton of $\tau^i \tilde{S}$ with the 1-skeleton of $\tau^j \tilde{S}$, $i \neq j$, then, again, we can only reduce possible intersections. So, we conclude that the sum of the complexities of the 2-spheres $\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_{k-1}$ is less than $k \cdot \mathcal{C}(S)$. Hence, some \tilde{S}_i must have $\mathcal{C}(\tilde{S}_i) < \mathcal{C}(\tilde{S})$. This is a contradiction to our choice of \tilde{S} .

We have shown for $\text{Fix}(G) = \emptyset$, the least weight normal 2-sphere \tilde{S} which is not in \mathcal{H} and has minimal complexity must actually have 0 complexity; and so, it is G -equivariant.

We now consider the case when $\text{Fix}(G) \neq \emptyset$.

As before, we can lift G to \tilde{G} acting on \tilde{M} and we can choose a least weight normal 2-sphere \tilde{S} not representing a class in \mathcal{H} with minimal complexity. Note we can again suppose for each $g \in \tilde{G}$, either $g(\tilde{S}) = \tilde{S}$, $g(\tilde{S})$ intersects \tilde{S} transversely, or $g(\tilde{S}) \cap \tilde{S} = \emptyset$. Also as \tilde{G} acts simplicially, $\text{Fix}(g)$ is a subcomplex of \mathcal{T} and so \tilde{S} is automatically transverse to $\text{Fix}(g)$.

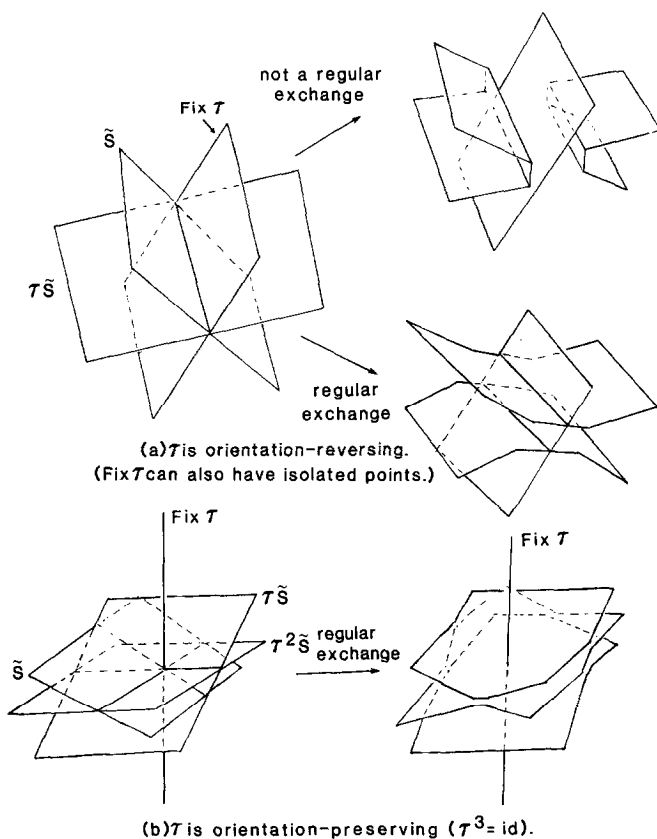


FIGURE 25

Assume \tilde{S} intersects $\tau\tilde{S}$ transversely, where τ is of infinite order in \tilde{G} . We can then apply the fixed-point-free argument, since a simplicial homeomorphism with fixed-points must be of finite order. In the case that τ is of finite order and \tilde{S} meets $\tau\tilde{S}$ transversely, we again consider the geometric sum (****). At a point where \tilde{S} intersects $\text{Fix}(\tau)$, there are two possibilities. (See Fig. 25. Note that as $\text{Fix}(\tau)$ is a subcomplex of $\tilde{\mathcal{P}}$, a regular exchange between \tilde{S} and $\tau\tilde{S}$ is uniquely specified, as in Fig. 25(a).) In both cases, exactly the same method as in the fixed-point-free situation works, and we get a contradiction to the choice of \tilde{S} as least weight.

If $\text{Fix}(G)$ is empty or not, there is a \tilde{G} -equivariant 2-sphere $\tilde{S} \subset \tilde{M}$, which is not in $\tilde{\mathcal{P}}$. The space M is the quotient space obtained by restricting to the subgroup of covering transformation, which is isomorphic to $\pi_1(M)$. Since covering transformations are fixed-point-free, we have that either $p: \tilde{S} \rightarrow M$ is an embedding or a 2-sheeted covering map onto an embedded projective plane P .

If $p: \tilde{S} \rightarrow M$ is an embedding, then $S = p\tilde{S}$ is a G -equivariant 2-sphere embedded in M and S does not represent a class in \mathcal{H} . This is the desired conclusion for Theorem 4.1.

So, suppose $p: \tilde{S} \rightarrow M$ is a 2-sheeted covering map onto an embedded projective plane P . If P is two-sided, then P is a G -equivariant two-sided projective plane embedded in M and P does not represent a class in \mathcal{H} . If P is not two-sided, then a 2-sphere S bounding a small equivariant regular neighborhood of P in M is a G -equivariant 2-sphere embedded in M ; and S does not represent a class in \mathcal{H} . In either situation we arrive at the desired conclusion for Theorem 4.1. ■

The classical Sphere Theorem [P₁, W, E] is an immediate corollary of the Equivariant Sphere Theorem. Our methods give a proof of the Sphere Theorem depending on the classical Loop Theorem and do not use a tower construction.

COROLLARY 4.2 (Sphere Theorem). *Let M be a 3-manifold and suppose \mathcal{H} is a proper $\pi_1(M)$ -invariant subgroup of $\pi_2(M)$. Then there is a 2-sphere or two-sided projective plane embedded in M , which does not represent an element of \mathcal{H} .*

Proof. If M is simply connected, then the proof is exactly the same as that of the claim in the proof of Theorem 4.1. If M is not simply connected, we consider the universal covering (\tilde{M}, p) of M . Let \mathcal{T} be a triangulation of M and lift \mathcal{T} to $\tilde{\mathcal{T}}$, a triangulation of \tilde{M} . Let \tilde{G} be the group of covering transformations and let $\tilde{\mathcal{H}} = p_*^{-1}\mathcal{H}$. Then \tilde{G} is a group of simplicial homeomorphisms and $\text{Fix}(\tilde{G}) = \emptyset$. Theorem 4.1 guarantees a \tilde{G} -equivariant 2-sphere \tilde{S} embedded in \tilde{M} and not in $\tilde{\mathcal{H}}$. We now obtain the desired 2-sphere or two-sided projective plane in M exactly as we finished the proof of Theorem 4.1. ■

THEOREM 4.3 (Equivariant Loop Theorem). *Let M be a 3-manifold with triangulation \mathcal{T} and suppose G is a group of simplicial homeomorphisms of M with $\text{Fix}(G)$ a subcomplex. Let B be a G -invariant collection of components of ∂M and suppose \mathcal{N} is a G -invariant normal subgroup of $\pi_1(B)$. Set $K = \ker(\pi_1(B) \rightarrow \pi_1(M))$. If $K - \mathcal{N} \neq \emptyset$, then there is a disk D embedded in M with $D \cap \partial M = D \cap B = \partial D$ and ∂D does not represent an element of \mathcal{N} .*

A 3-manifold M is P^2 -irreducible if it contains no two-sided projective planes and each 2-sphere in M bounds a 3-cell in M . If any covering space (\tilde{M}, p) of a 3-manifold M is P^2 -irreducible, then M is itself P^2 -irreducible. The next theorem, which provides the converse to the preceding obser-

vation, was proved by Meeks, Simon, and Yau in [M-S-Y], using least area surfaces.

THEOREM 4.4. *Let M be a P^2 -irreducible 3-manifold. Let (\tilde{M}, p) be the universal covering of M . Then M is P^2 -irreducible.*

Proof. If \tilde{M} were not P^2 -irreducible, then there would be a 2-sphere in \tilde{M} , which did not bound a 3-cell in \tilde{M} . Since \tilde{M} is simply connected, we do not need to be concerned with any possible projective planes embedded in \tilde{M} . Let \mathcal{T} be a triangulation of M and lift \mathcal{T} to a triangulation $\tilde{\mathcal{T}}$ of \tilde{M} . Let $\tilde{G} \approx \pi_1(M)$ be the group of covering transformations. Then \tilde{G} is a group of simplicial homeomorphisms with $\text{Fix}(\tilde{G}) = \emptyset$. If \mathcal{H} is the set of 2-spheres in \tilde{M} which bound 3-cells in \tilde{M} , then by supposition there is a 2-sphere not in \mathcal{H} . By Theorem 2.2 there is a least weight normal 2-sphere \tilde{S} not in \mathcal{H} . Using the same argument as that in the proof of Theorem 4.1, we can establish that if \tilde{S} also has minimal complexity with respect to \tilde{G} , then \tilde{S} is \tilde{G} -equivariant. Since $\text{Fix}(\tilde{G}) = \emptyset$, $p: \tilde{S} \rightarrow M$ is either an embedding or a 2-sheeted covering map onto a projective plane P . If p is an embedding, then $S = p(\tilde{S})$ is a 2-sphere which does not bound a 3-cell in M . If $p: \tilde{S} \rightarrow P$ is a 2-sheeted covering, then if P is two-sided, M contains a two-sided projective plane, and if P is not two-sided, the boundary of a regular neighborhood of P in M is a 2-sphere in M which does not bound a 3-cell in M . Any one of these possibilities contradicts the assumption that M is P^2 -irreducible. ■

THEOREM 4.5. *Let M be an irreducible and ∂ -irreducible 3-manifold with triangulation \mathcal{T} and let G be a finite group of simplicial homeomorphism of M with $\text{Fix}(G)$ a subcomplex. Suppose F is a compact incompressible and ∂ -incompressible surface (not necessarily connected) in M having the property that each component of the G -orbit of F is homotopically disjoint from F . Then there is a surface F' in M , isotopic to F , such that F' is G -equivariant.*

Proof. From the hypotheses on M and F , it follows from Theorem 2.4 that F is isotopic to a least weight normal surface. We shall continue to use F for this surface. Notice that we maintain the property that each component of the G -orbit of F is homotopically disjoint from F . We may assume for $g \in G$, either $g(F) \cap F = \emptyset$, $g(F) = F$, or $g(F)$ and F meet transversely.

If two components F_1 and F_2 in the G -orbit of F meet transversely, then by Theorem 3.7 each curve of the intersection $F_1 \cap F_2$ is a regular curve of intersection and the result of regular exchanges at each curve of intersection gives disjoint surfaces F'_1 and F'_2 , where F'_i is isotopic to F_i , $i = 1, 2$.

Now the entire G -orbit of F consists of a finite number of surfaces. Therefore, we can make regular exchanges in all possible curves of intersection (the order does not matter) and we obtain a G -invariant collection of disjoint surfaces. Fortunately, we know what the result of these regular exchanges gives us. Again by Theorem 3.7, we have a surface F' , isotopic with F , along with the entire G -orbit of F' . Hence, F' is the desired G -equivariant surface. ■

5. INVARIANT DECOMPOSITIONS

In this section we show our methods extend to the generalized versions of the Loop Theorem, where collections of disks are considered, and the Sphere Theorem, where collections of 2-spheres and two-sided projective planes are considered. In both cases we obtain special G -invariant collections. Similar results appear in [M-Y₁] and [M-S].

For prime decompositions of 3-manifolds, where no factor is a 2-sphere bundle over S^1 , we are able to get a G -invariant prime decomposition (see also [M-Y₁]). Similarly, we show, with a possible exception, that one can choose a G -invariant characteristic submanifold of a ∂ -incompressible Haken-manifold. The one exception is when the manifold under consideration is a torus bundle over S^1 and is not Seifert fibered (i.e., a torus bundle over S^1 where a neighborhood of the fiber is characteristic) [M-S].

THEOREM 5.1. *Let M be a compact 3-manifold with triangulation \mathcal{T} and let G be a finite group of simplicial homeomorphisms of M with $\text{Fix}(G)$ a subcomplex. Then there is a finite, G -invariant collection of pairwise disjoint and embedded 2-spheres and two-sided projective planes which generate $\pi_2(M)$ as a $\pi_1(M)$ -module.*

Proof. We may as well assume $\pi_2(M) \neq 0$. By Theorem 4.1 there is a G -equivariant, embedded 2-sphere or two-sided projective plane in M , which represents a nontrivial class of $\pi_2(M)$. Let $\Sigma_1, \dots, \Sigma_k$ denote the G -orbit of this surface. The collection $\Sigma_1, \dots, \Sigma_k$ is G -invariant. If the $\pi_1(M)$ -submodule of $\pi_2(M)$ generated by the collection $\Sigma_1, \dots, \Sigma_k$ is $\pi_2(M)$, then we are done. Otherwise, we show we can add to the collection $\Sigma_1, \dots, \Sigma_k$ and eventually generate $\pi_2(M)$ as a $\pi_1(M)$ -module.

So, we suppose we have a finite, G -invariant collection $\Sigma_1, \dots, \Sigma_k, \dots, \Sigma_n$ of pairwise disjoint and embedded 2-spheres and two-sided projective planes in M .

Let \mathcal{H} denote the $\pi_1(M)$ -submodule of $\pi_2(M)$ generated by $\Sigma_1, \dots, \Sigma_n$.

Then \mathcal{H} is G -invariant. If \mathcal{H} is not proper, then we are done: so, suppose \mathcal{H} is a proper $\pi_1(M)$ - and G -invariant submodule of $\pi_2(M)$.

By Theorem 4.1 there is a G -equivariant, embedded 2-sphere or two-sided projective plane S in M so that the class of S is not in \mathcal{H} .

If $S \cap \Sigma_i = \emptyset$ for $1 \leq i \leq n$, then the G -orbit of S is disjoint from Σ_i for $1 \leq i \leq n$. So the G -orbit of S , along with $\Sigma_1, \dots, \Sigma_n$, is a collection of pairwise disjoint and embedded 2-spheres and two-sided projective planes, which contains $\Sigma_1, \dots, \Sigma_n$ as a proper subcollection and generates a $\pi_1(M)$ -submodule of $\pi_2(M)$ containing \mathcal{H} as a proper submodule.

If S meets some Σ_i , we may assume notation has been chosen so that $S \cap \Sigma_1 \neq \emptyset$. In addition, we may assume this intersection is transverse. Let S_1, \dots, S_m denote the components of the G -orbit of S .

Since all projective planes under consideration are two-sided and intersections are transverse, a curve of intersection between any S_i and any Σ_j must be a contractible curve in each. Furthermore, if we restrict to any Σ_j , then the intersection of the entire collection S_1, \dots, S_m with Σ_j is a collection of pairwise disjoint simple closed curves. So, in particular, the intersection of the collection S_1, \dots, S_m with Σ_1 is a collection of pairwise disjoint simple closed curves.

Let $J \subset (S_1 \cup \dots \cup S_m) \cap \Sigma_1$ be an innermost curve on Σ_1 and suppose $J \subset \Sigma_1 \cap S_1$. Then J bounds a disk $D \subset \Sigma_1$, where $\text{int}(D) \cap S_i = \emptyset$, $1 \leq i \leq m$. Let E and E' denote the closures of the component of $S_1 - J$. Both E and E' are disks, or one is a disk and one is a Moebius band.

Since S_1 does not represent a class in \mathcal{H} (\mathcal{H} is G -invariant), either $D \cup E$ or $D \cup E'$ does not represent a class in \mathcal{H} . Suppose notation has been chosen so that $D \cup E'$ does not represent a class in \mathcal{H} . Let D' denote an equivariant "shove off" of D in a neighborhood of Σ_1 . Then the surface $S' = D' \cup E'$ is an embedded 2-sphere or two-sided projective plane in M such that the class of S' is not in \mathcal{H} (note that S' is not necessarily G -equivariant). Furthermore, if $S'_1, \dots, S'_{m'}$ (m' may not be the same as m) denotes the G -orbit of S' , then the total number of curves in the intersection of Σ_1 (and hence Σ_i) with $S'_1, \dots, S'_{m'}$ has been reduced. This procedure eventually yields an embedded 2-sphere or two-sided projective plane S'' in M , for which the G -orbit of S'' is disjoint from all Σ_i , $1 \leq i \leq m$, and S'' does not represent a class in \mathcal{H} . But then we can apply the method of Theorem 4.1, using least complexity normal 2-spheres of least weight in \tilde{M} which do not intersect any of the lifts of the Σ_i to \tilde{M} , $1 \leq i \leq m$, and which represent classes not in \mathcal{H} . In this fashion, we can find a G -invariant collection of pairwise disjoint and embedded 2-spheres and two-sided projective planes which contains $\Sigma_1, \dots, \Sigma_n$ as a proper subcollection and generates a $\pi_1(M)$ -submodule of $\pi_2(M)$ containing \mathcal{H} as a proper submodule.

In both possibilities, we obtain a G -invariant collection $\Sigma_1, \dots, \Sigma_n, \Sigma_{n+1}, \dots, \Sigma_N$ of pairwise disjoint, embedded 2-spheres and two-sided pro-

jective planes containing the collection $\Sigma_1, \dots, \Sigma_n$ as a proper subcollection and generating a $\pi_1(M)$ - and G -invariant submodule of $\pi_2(M)$ containing \mathcal{H} as a proper submodule.

By the Finiteness Theorem of Kneser [K, H₂] (also see Theorem III.20 of [J₁]), there is an N (dependent only on \mathcal{T}) so that the collection $\Sigma_1, \dots, \Sigma_N$ must generate $\pi_2(M)$ as a $\pi_1(M)$ -module. ■

The next theorem is the Loop Theorem version of the preceding theorem [M-Y₂]. Its proof can be obtained using the same ideas as in the proof of Theorem 5.1 but applying Theorem 4.3 of Section 4.

THEOREM 5.2. *Let M be a compact 3-manifold with triangulation \mathcal{T} and let G be a finite group of simplicial homeomorphisms of M with $\text{Fix}(G)$ a subcomplex. Suppose B is a collection of components of ∂M , which is G -invariant. Set $K = \ker(\pi_1(B) \rightarrow \pi_1(M))$. Then there is a G -invariant collection of pairwise disjoint, properly embedded disks D_1, \dots, D_n in M such that the curves $\partial D_1, \dots, \partial D_n$ represent nontrivial classes in $\pi_1(B)$ and normally generate K .*

The collection of 2-spheres and two-sided projective planes in the conclusion to Theorem 5.1 may not be minimal with respect to generating $\pi_2(M)$ as a $\pi_1(M)$ -module. A similar remark applies to the collection of disks in the conclusion to Theorem 5.2 with respect to the normal generation of $K = \ker(\pi_1(B) \rightarrow \pi_1(M))$. Of course, in both cases a subcollection can be selected which is minimal with respect to generation; however, in general, we must replace G -invariant by G -equivariant, if we desire such a minimal subcollection. The next two theorems do allow for a sharp conclusion with respect to the minimality of the collection and its G -invariance. First, we need a couple of definitions.

The 3-manifold P is *prime* if P is not homeomorphic to S^3 and whenever $P = Q \# N$, either Q or N is homeomorphic to S^3 . A compact 3-manifold M can be expressed as

$$M = P_1 \# \dots \# P_n, \quad (*****)$$

where each P_i is prime [K]; furthermore, if M is orientable, or no P_i is homeomorphic to a 2-sphere bundle over S^1 , the decomposition (*****) is unique up to homeomorphism and order of factors [M]. We call the decomposition (*****) a *prime decomposition* of M .

THEOREM 5.3. *Let M be a compact 3-manifold with triangulation \mathcal{T} and let G be a finite group of simplicial homeomorphisms of M with $\text{Fix}(G)$ a subcomplex. If the prime decomposition of M does not contain a 2-sphere bundle over S^1 , then there is a G -invariant, prime decomposition of M .*

Proof. Let $M = P_1 \# \cdots \# P_n$ be a prime decomposition of M . There are $(n-1)$ 2-spheres $\Sigma_1, \dots, \Sigma_{n-1}$ embedded in M which determine this collection of prime summands. We say the 2-spheres $\Sigma_1, \dots, \Sigma_{n-1}$ *determine a prime decomposition of M* . We wish to find a G -invariant collection of 2-spheres which determine a prime decomposition of M .

Claim (Existence). If M admits a prime decomposition, which does not contain any 2-sphere bundles over S^1 , then for any triangulation \mathcal{T} of M there is a collection of least weight normal 2-spheres which determine a prime decomposition of M .

Standard techniques going back to [K] provide for normalization of such a collection. After normalization we take a collection having least weight among all collections of normal 2-spheres, which determine a prime decomposition of M .

Suppose $\Gamma_1, \dots, \Gamma_k, A_{k+1}, \dots, A_{n-1}$ and $\Sigma_1, \dots, \Sigma_k, A_{k+1}, \dots, A_{n-1}$ are two least weight collections of normal 2-spheres, each of which determines a prime decomposition of M . (We have written these collections in this fashion since it may be the case that the collections contain some of the same 2-spheres. If this is not the case, then $k = n-1$.) Furthermore, suppose for all pairs i, j , $1 \leq i, j \leq k$, we have $\Gamma_i \cap \Sigma_j = \emptyset$ or Γ_i meets Σ_j transversely.

Claim. (1) Every curve of $\Gamma_i \cap \Sigma_j$, $1 \leq i, j \leq k$, is a regular curve of intersection.

(2) The result of regular exchanges at each curve of $\Gamma_i \cap \Sigma_j$, $1 \leq i, j \leq k$, is a pairwise disjoint collection of 2-spheres $\Gamma'_1, \dots, \Gamma'_k, \Sigma'_1, \dots, \Sigma'_k$ where $\text{wt}(\Gamma'_i) = \text{wt}(\Gamma_i)$, $1 \leq i \leq k$, $\text{wt}(\Sigma'_j) = \text{wt}(\Sigma_j)$, $1 \leq j \leq k$.

(3) The collections $\Gamma'_1, \dots, \Gamma'_k, A_{k+1}, \dots, A_{n-1}$ and $\Sigma'_1, \dots, \Sigma'_k, A_{k+1}, \dots, A_{n-1}$ are both least weight collections of normal 2-spheres, which determine prime decompositions of M .

The proof of this claim does not require the detailed combinatorics required in the proof of Proposition 3.1. It can be argued as we did in the proof of Proposition 3.4.

So, begin with a least weight collection of embedded normal 2-spheres in M , which determine a prime decomposition of M . We consider its orbit under G and apply the above claim. The conclusion is that there is a least weight collection of embedded normal 2-spheres, which determines a prime decomposition of M and is G -equivariant.

We wish to find a G -invariant prime decomposition. Our proof is via induction on the number of prime summands. If M is prime (one prime summand), the theorem is vacuously satisfied. So, suppose we can find the

desired prime decomposition for manifolds having fewer than n factors, $n > 1$. Let M be a 3-manifold having a prime decomposition with n factors. Let $\Sigma_1, \dots, \Sigma_{n-1}$ be a G -equivariant collection of 2-spheres which determine a prime decomposition of M . Since each Σ_i , $1 \leq i \leq n-1$, separates M , there is a prime summand, say P_n , which meets only one of the 2-spheres $\Sigma_1, \dots, \Sigma_{n-1}$, say, P_n meets Σ_{n-1} .

It follows that if $g\Sigma_j$ is in the G -orbit of Σ_j and $g\Sigma_j \cap P_n \neq \emptyset$, then $g\Sigma_j = \Sigma_{n-1}$ or $g\Sigma_j$ is parallel in P_n to Σ_{n-1} . Hence the G -orbit of the spheres $\Sigma_1, \dots, \Sigma_{n-1}$ meets P_n in a product neighborhood of Σ_{n-1} in P_n . Denote the innermost (in P_n) one of these spheres by A and the subregion of P_n determined by A as P'_n . The G -orbit of P'_n is made up of prime summands of M and is G -invariant. We obtain a new G -equivariant collection of 2-spheres which determine a prime decomposition of M by replacing each Σ_i , which is parallel in M to a gA , by the 2-sphere gA . This works, except in the special case $n=2$, where the entire G -orbit of Σ_1 is parallel to Σ_1 (since G is finite the orbit consists of Σ_1 and at most one other 2-sphere). In this special case we apply the work of [M-Y₂] to finite group actions on $S^2 \times 1$ to obtain the desired G -invariant 2-sphere. Now, replacing P'_n and the G -orbit of P'_n by a 3-cell, we obtain a 3-manifold which G acts on and which has a prime decomposition with fewer than n summands. The induction provides a G -invariant prime decomposition of this manifold. This decomposition along with the G -orbit of A is the desired G -invariant prime decomposition of M . ■

THEOREM 5.4. *Let M be a compact, orientable, ∂ -irreducible Haken-manifold with triangulation \mathcal{T} . Suppose G is a finite group of simplicial homeomorphisms of M with $\text{Fix}(G)$ a subcomplex. Then the characteristic submanifold of M can be chosen to be G -invariant, or M is a torus bundle over S^1 and $V(M)$ is a neighborhood of the fiber.*

Proof. The idea of this proof is very similar to the proof of Theorem 5.3; here, we first find a characteristic submanifold with G -equivariant frontier. Then we find a G -equivariant characteristic submanifold. From a G -equivariant characteristic submanifold, we are able to obtain a G -invariant one or M must be a torus bundle over S^1 and $V(M)$ is a neighborhood of the fiber.

Recall that the characteristic submanifold of a ∂ -irreducible Haken-manifold is unique up to isotopy [Jo, J-S]. Also, if $V(M)$ is characteristic in M , then $\text{fr}(V(M))$, the frontier of $V(M)$, consists of properly embedded incompressible and ∂ -incompressible, essential annuli and tori. It follows for $V(M)$ and $V'(M)$ both characteristic, then $\text{fr}(V(M))$ is homotopically disjoint from $\text{fr}(V'(M))$. Now, by Theorem 4.5, there is a characteristic submanifold $V'(M)$ of M such that $\text{fr}(V'(M))$ is G -equivariant.

The proof that there is a G -equivariant characteristic submanifold and the completion of the argument can be carried out as in the proof of Theorem 8.6 of [M-S]. ■

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